

Non-Uniform Compressive Sensing

Nazanin Rahnavard, Ali Talari, and Behzad Shahrasi

School of Electrical and Computer Engineering
Oklahoma State University, Stillwater, OK 74078

Emails: {nazanin.rahnavaard, ali.talari, behzad.shahrasi}@okstate.edu

Abstract—In this paper, we introduce *non-uniform compressive sensing* (NCS), a novel compressive sensing (CS) technique that enables non-uniform recovery of sparse signals coefficients from their undersampled random projections. This is in contrast to the conventional CS techniques and is of great interest in many practical applications in which coefficients of a signal may have unequal importance levels. We design NCS by modifying the CS sampling phase and our simulation results indicate that by properly tuning the parameters of NCS, desired non-uniform recovery is attainable. The coefficients with more importance are recovered with a higher probability compared to uniform CS, in return for a slight performance loss in the recovery of less important coefficients. To make the theoretical analysis of NCS tractable, we further study NCS under a simple decoder and confirm that desired non-uniform recovery can be achieved by NCS.

Moreover, we demonstrate that NCS can provide unequal recovery time. This means that different parts of signal \underline{x} can reach a target reconstruction quality at different time spans from the beginning of the measurement stream. NCS is mainly inspired by a previous work by Rahnavard et al. on unequal-error-protection rateless codes.

I. INTRODUCTION

Emerging *compressive sensing* (CS) techniques [1, 2] provide means to recover a *compressible* signal from its undersampled *random projections* also called *measurements*. Let $\underline{\theta} \in \mathbb{R}^n$ be the representation of signal $\underline{x} \in \mathbb{R}^n$ ($\underline{x} = [x_1 x_2 \dots x_n]^T$) in transform basis $\Psi \in \mathbb{R}^{n \times n}$, i.e., $\underline{x} = \Psi \underline{\theta}$. It is said that \underline{x} is *compressible* in Ψ if the magnitude of the coefficients of $\underline{\theta}$, after being sorted based on their absolute value, decay faster than $ci^{-\frac{1}{\tau}}$, where $0 < \tau \leq 1$ and c is a constant [2–5]. A larger τ shows a higher compressibility. Similar to [5], we can also make $\underline{\theta}$ sparse by keeping its $k \ll n$ significant coefficients and setting the rest of $n - k$ coefficients to zero. Such

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a signal is referred to as a k -sparse signal. CS is comprised of the two following key steps:

Signal Sampling: The random projections (measurements) are generated by $\underline{y} = \Phi \underline{x}$, where $\Phi \in \mathbb{R}^{m \times n}$ is a well-chosen random matrix, called *projection matrix* (aka measurement matrix), and $\underline{y} \in \mathbb{R}^m$ is the measurement vector. We can see that the i^{th} measurement, $y_i = \sum_{j=1}^n \varphi_{i,j} x_j$, where $\varphi_{i,j}$ is the entry on the i^{th} row and j^{th} column of Φ .

Signal Recovery: Signal reconstruction can be done by obtaining the estimate $\hat{\underline{\theta}}$ (and accordingly $\hat{\underline{x}} = \Psi \hat{\underline{\theta}}$) from the system of linear equations $\underline{y} = \Phi \Psi \underline{\theta}$. This is an underdetermined system with infinitely many solutions. However, the knowledge of $\underline{\theta}$ being a sparse signal allows us to have a successful reconstruction with high probability from $m = O(k \log n)$ measurements by solving the ℓ_1 optimization problem given by [1–3, 6]

$$\hat{\underline{\theta}} = \underset{\underline{\theta}}{\operatorname{argmin}} \|\underline{\theta}\|_1, \text{ s.t. } \underline{y} = \Phi \Psi \underline{\theta}, \quad (1)$$

where $\|\underline{\theta}\|_1 = \sum_{i=1}^n |\theta_i|$. The ℓ_1 optimization problem (1) can be solved with linear programming techniques such as *basis pursuit* (BP) [6].

To the best of our knowledge, all existing CS studies have merely considered *equal importance* for all the coefficients of \underline{x} . Therefore, all x_i 's are equally treated and are recovered with the same probability in the recovery phase. However, in many practical applications signal coefficients may have *unequal importance* levels. For instance, if \underline{x} represents the temperature readings of a sensor network deployed in a field, we may be more interested in monitoring a certain area's temperature with a higher accuracy.

In this paper, we propose *non-uniform compressive sensing* (NCS) for non-uniform recovery of signal coefficients based on their importance levels. Our approach is based on a novel design for the projection matrix Φ such that the measurements capture more important coefficients with higher probabilities. In

NCS, the decoding can remain intact and conventional CS reconstruction algorithms such as BP may be employed. To design NCS, we adopt ideas from the previous work on *rateless codes with unequal-error-protection* (UEP) property [7, 8] in which the encoding phase of rateless codes is modified to provide UEP.

This paper is organized as follows. In Section II, we review the related work. In Section III, we propose NCS and evaluate its performance and properties by extensive simulations. In Section IV, we theoretically analyze NCS for a case with a simple and suboptimal recovery scheme. Finally, Section V concludes the paper.

II. RELATED WORK

A great portion of the work on CS has been devoted to enhance the performance of signal recovery by introducing new reconstruction algorithms [9–16]. However, the signal sampling process in CS and the type of projection matrices that are employed have not evolved as much.

In the early work on CS [2, 17], the measurement matrices are chosen to be dense matrices with entries selected independently from some distribution such as Bernoulli or Gaussian. Sparse measurement matrices have also been proposed as they provide low encoding and decoding complexity [12, 18–20]. Such sparse measurement matrices are shown to perform almost as good as dense Gaussian and Fourier matrices under certain conditions.

However, the aforementioned work can only provide *uniform* reconstruction of signal's coefficients. Further, these CS schemes mostly focus on signal reconstruction phase rather than signal sampling as in NCS. To the best of our knowledge this work is the first to propose a novel design for projection matrices to provide non-uniform recovery of signal coefficients.

III. NCS DESIGN

Following rateless codes [21], we may view the coefficients of \underline{x} and the measurements \underline{y} in a CS scheme as vertices of a bipartite graph G , where coefficients are *information nodes* and measurements are *encoded nodes*. The coefficient x_i is connected to measurement y_j with an edge of weight $\varphi_{j,i}$.

Conventionally, the measurement matrix Φ has been considered to be a dense matrix with its entries chosen randomly from $\mathcal{N}(0, 1)$ or $\{+1, -1\}$ with equal probabilities. For a dense Φ , all measurements have edges

connected to all coefficients. Hence, all coefficients would have degree (the number of edges connected to a node) m and all measurements would have degree n . Therefore, all coefficients are included in all measurements and the probability of their recovery will be the same. Consequently, a dense Φ may only provide *uniform* protection for all coefficients. Figure 1 shows G when Φ is a dense matrix.

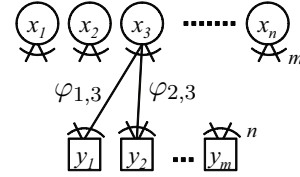


Fig. 1. Bipartite graph representing the sampling phase in a CS scheme. Circles and squares represent coefficients \underline{x} and measurements \underline{y} , respectively, where $\underline{y} = \Phi \underline{x}$.

To design *non-uniform CS*, we employ the ideas from UEP rateless codes [7, 8]. In UEP rateless coding, the encoded nodes in the bipartite graph have more edges connected to more important information nodes. Therefore, more important information nodes are included in more encoded nodes and the probability that their decoding is successful increases. We employ the same idea and propose to incorporate more important coefficients in a larger number of measurements or equivalently sample them with a higher frequency. This leads to non-uniform distribution of edges over the information nodes (coefficients) in G . Clearly, this can be possible only if the measurement matrix is *sparse*. Therefore, similar to [5, 12, 18, 19], we consider a sparse Φ with $L \ll n$ nonzero entries per row selected from $\{+\sqrt{\frac{n}{L}}, -\sqrt{\frac{n}{L}}\}$ with equal probabilities. To infuse non-uniform recovery, these non-zero entries need to be non-uniformly distributed over each row of Φ .

Let us partition n coefficients of \underline{x} into r sets s_1, s_2, \dots, s_r with decreasing importance levels and sizes $\alpha_1 n, \alpha_2 n, \dots, \alpha_r n$ such that $\sum_{i=1}^r \alpha_i = 1$. Let $p_j(n)$ ¹ be the probability that an edge emanating from a measurement is connected² to a particular coefficient in s_j , for $j = 1, 2, \dots, r$. Clearly, $\sum_{j=1}^r p_j \alpha_j n = 1$.

By setting appropriate values for p_j 's we can adjust the frequency that measurements capture coefficients

¹The special case $p_1 = \dots = p_r = \frac{1}{n}$, results in the previously studied uniform CS.

²Parallel edges could occur but asymptotically its probability is negligible.

in s_j 's and provide non-uniform recovery. Such an ensemble of NCS is specified by parameters $L, n, m, \underline{\alpha} = \{\alpha_1, \dots, \alpha_r\}$, and $\underline{p} = \{p_1, \dots, p_r\}$. Figure 2 depicts how non-uniform selection of coefficients in bipartite graph G is performed.

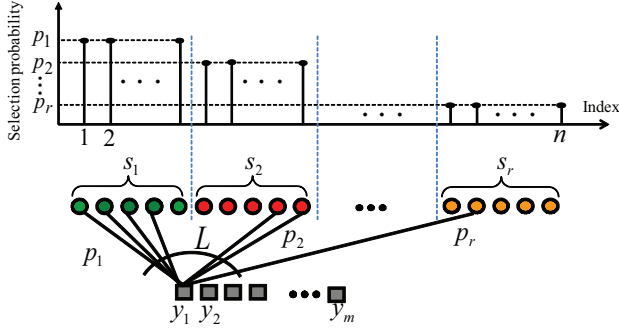


Fig. 2. Non-uniform selection of signal coefficients by measurements in NCS.

Clearly, the higher p_j is chosen, the more coefficients from s_j are contributing to a measurement. Therefore, the resulting measurement matrix $\Phi_U = [\Phi^1 | \Phi^2 | \dots | \Phi^r]$ is comprised of r sub-matrices Φ^j of size $m \times \alpha_j n$, for $j \in \{1, 2, \dots, r\}$. An entry in Φ^j is non-zero with probability Lp_j . Hence, a higher value of p_j maps to a higher density of non-zero entries in Φ^j . Figure 3 shows the structure of Φ_U .

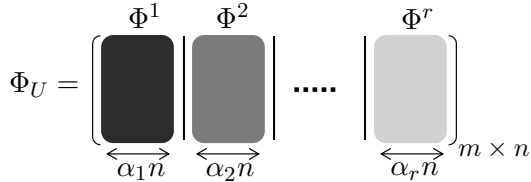


Fig. 3. Submatrices of Φ_U in NCS. A darker color corresponds to a denser matrix.

In the proposed NCS, only the sampling phase has been changed compared to conventional CS schemes and the decoding has not been altered. However, the question that arises here is, whether employing Φ_U satisfies CS requirements similar to a dense Φ . Authors in [5] showed that for signals that are sparse in Fourier transform basis with $\varphi_{i,j}$'s satisfying

$$E[\varphi_{i,j}] = 0, E[\varphi_{i,j}^2] = 1, \text{ and } E[\varphi_{i,j}^4] = \frac{n}{L}, \quad (2)$$

and

$$L \geq \begin{cases} 1 & \text{if } 0 < \tau < 1, \\ \log n & \text{if } \tau = 1. \end{cases} \quad (3)$$

the CS reconstruction employing a sparse Φ results in almost the same reconstruction accuracy of a dense

Φ . Further, [22] showed that sparse Φ matrices can be employed with any orthonormal dense Ψ . Consequently, we only need to show that $\varphi_{i,j}$'s of Φ_U satisfy (2) and select L to satisfy (3).

Lemma 1: Elements of a measurement matrix $\Phi_U = [\Phi^1 | \Phi^2 | \dots | \Phi^r]$ of NCS with parameters $L, n, m, r, \underline{\alpha}$, and \underline{p} satisfy (2).

Proof: We note that each entry of sub-matrix $\Phi^j = [\phi^j]$ is zero, $+\sqrt{\frac{n}{L}}$, or $-\sqrt{\frac{n}{L}}$ with probabilities $1 - Lp_j$, $Lp_j/2$, and $Lp_j/2$, respectively. For $q \in \{1, 2, \dots, n\}$ and $q_j \in \{1, 2, \dots, \alpha_j n\}$ we have

$$E[\phi_{i,q}] = \sum_{j=1}^r \alpha_j E[\phi_{i,q_j}^j] = 0,$$

$$E[\phi_{i,q}^2] = \sum_{j=1}^r \alpha_j E[(\phi_{i,q_j}^j)^2] = \sum_{j=1}^r \alpha_j n p_j = 1,$$

$$\begin{aligned} E[\phi_{i,q}^4] &= \sum_{j=1}^r \alpha_j \left(\frac{Lp_j}{2} \left(\frac{n}{L} \right)^2 + \frac{Lp_j}{2} \left(\frac{n}{L} \right)^2 \right) \\ &= \frac{n^2}{L} \sum_{j=1}^r \alpha_j p_j = \frac{n}{L}. \end{aligned}$$

Thus, the entries of Φ_U satisfy (2). ■

In the next section, we evaluate the properties of NCS.

A. Performance Evaluation of NCS

In this section, we consider a special case of NCS in which a signal \underline{x} has two importance levels ($r = 2$) and compare the performance of NCS to the performance of uniform CS (in which L non-zero elements in each row of Φ have been distributed uniformly). Assume $n_1 = \alpha n$ is the number of *more important coefficients* (MICs), which reside in the first part of \underline{x} , and $n_2 = (1 - \alpha)n$ is the number of *less important coefficients* (LICs). To infuse non-uniform recovery, we set $p_1 = \frac{k_M}{n}$ and $p_2 = \frac{k_L}{n}$ for some $0 < k_L \leq 1$ and $k_M = \frac{1 - (1 - \alpha)k_L}{\alpha}$. This ensures $p_1 \geq p_2$ (higher recovery for MICs).

For our simulations, we assume $\alpha = 0.15$, \underline{x} has length $n = 10^3$, and it is a k -sparse signal in DCT domain with $k = 10^2$. Therefore, we randomly generate $\underline{\theta}$ with $k = 10^2$ nonzero entries with $\tau = \frac{7}{8}$ [23] and find \underline{x} from $\underline{x} = \Psi \underline{\theta}$. With this value of τ , we have $L \geq 1$ from (3). In our simulations, we will show that the performance is indeed insensitive to the exact value of L and we set $L = 100$.

We perform the CS sampling (encoding) employing Φ_U and perform the reconstruction (decoding) employing the conventional basis pursuit CS reconstruction to obtain an estimate of the encoded signal \hat{x} . We plot the *normalized reconstruction error* (NRE) of MICs and LICs given by $\frac{\|\underline{x}_{MIC} - \hat{x}_{MIC}\|_2}{\|\underline{x}_{MIC}\|_2}$ and $\frac{\|\underline{x}_{LIC} - \hat{x}_{LIC}\|_2}{\|\underline{x}_{LIC}\|_2}$ versus the simulation parameters, where $\underline{x}_{MIC} = [x_1 x_2 \dots x_{\alpha m}]^T$ and $\underline{x}_{LIC} = [x_{\alpha n+1} x_{\alpha n+2} \dots x_n]^T$. In the first simulation, we fix the number of measurements to $m = 300$ and plot NRE of MICs and LICs versus k_M as illustrated in Figure 4. We should note that $k_M = 1$ corresponds to uniform CS.

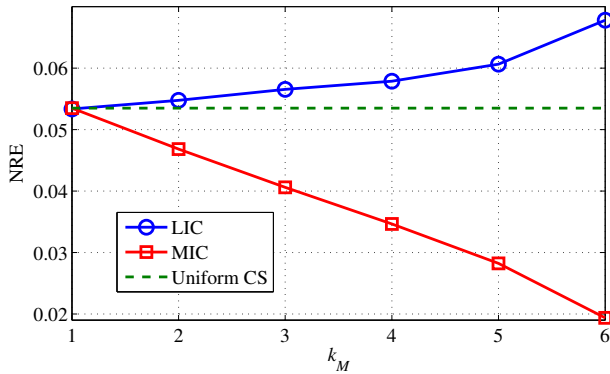


Fig. 4. NREs for MICs and LICs versus k_M . Note that $k_M = 1$ corresponds to the NRE of uniform CS.

Figure 4 shows that as k_M increases MICs are recovered with a higher accuracy compared to LICs. Therefore, the desired non-uniform recovery has been provided for MICs and LICs. Moreover, the non-uniformity can be easily adjusted by appropriately setting k_M . As an example, we observe that at $k_M = 4$, MICs are recovered with 35% higher accuracy compared to uniform CS ($k_M = 1$), while the performance loss of LICs is less than 10%.

Next, we fix $k_M = 4$ and plot NREs of MICs and LICs versus the number of measurements m as illustrated in Figure 5. We have also included NRE for uniform CS ($k_M = 1$).

Figure 5 can be interpreted in two ways. First, we can assume the number of measurements is fixed. Hence, for a given m we see that MICs have been recovered with a much better accuracy compared to the uniform CS, while there is a slight performance loss in the reconstruction of LICs. On the other hand, we may target for a fixed NRE and observe that MICs achieve the target NRE with fewer measurements. To

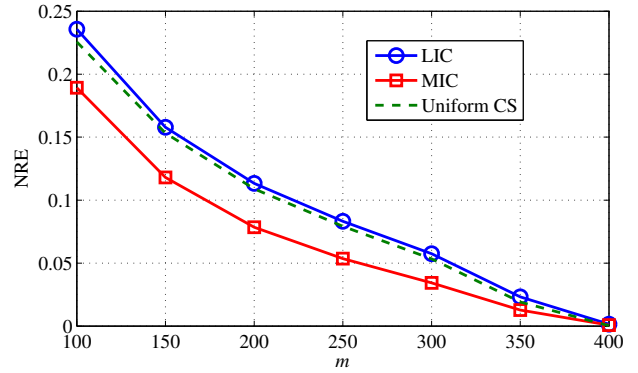


Fig. 5. NRE versus m for NCS. The dashed line corresponds to the NRE of uniform CS.

elaborate more on this property, we have also plotted the number of measurements required to achieve the target error rate of $NRE=0.1$ for MICs and LICs versus k_M in Figure 6. Figure 6 shows that MICs achieve $NRE=0.1$ with about 20% less number of measurements than LIC. This can actually provide *unequal recovery time* (URT). This means that different parts of signal \underline{x} can reach a target reconstruction quality at different time spans from the beginning of the measurement stream.

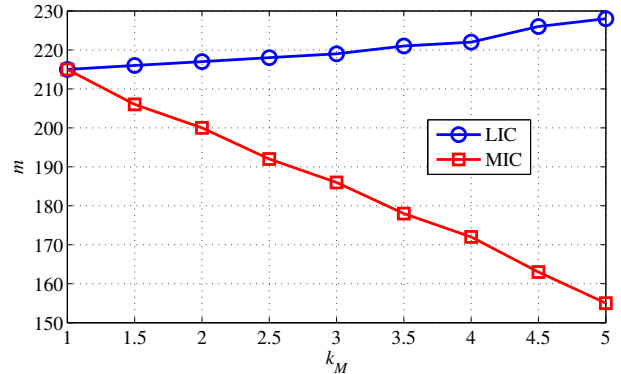
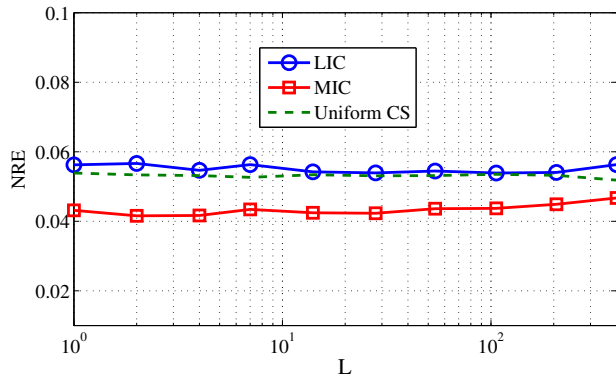


Fig. 6. The number of required measurements m versus k_M for MICs and LICs to reach $NRE=0.1$. Clearly, $k_M = 1$ corresponds to uniform CS.

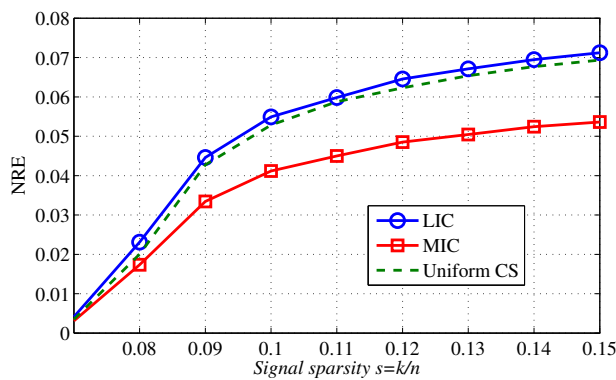
In Figure 7(a), we plot NREs versus the row weight L when $k_M = 2.5$ and $m = 300$. We observe that NREs are almost insensitive to row weight L and the reconstruction error remains almost constant.

In Figure 7(b), we plot NREs versus the sparsity $s = \frac{k}{n}$, by varying k . We observe that NRE is an increasing function of s . This is expected since increasing s results in a less compressible (less sparse) signal. However, we observe that non-uniform recov-

ery is still effectively provided for MICs and LICs compared to uniform CS.



(a) Plot of NRE versus row weight L .



(b) Plot of NRE versus sparsity of $s = \frac{k}{n}$.

Fig. 7. NRE versus L and $s = \frac{k}{n}$. Dashed line corresponds to the NRE of uniform CS.

In the next section, to make the asymptotic analysis of NCS tractable, we assume a simple iterative decoder is employed at the reconstruction phase.

IV. ANALYSIS OF NCS BASED ON A SIMPLE DECODER

In this section, we consider a simple *iterative* decoder based on *group-testing* CS reconstruction algorithms [18, 24] referred to by Decoder-I¹. In group-testing based CS reconstruction algorithms it is assumed that \underline{x} is k -sparse ($\Psi = I_n$). Similar to [18, 24], we assume that $L = \frac{n}{k}$. Since \underline{x} is sparse, there is a high probability that measurements capture L zero valued coefficients and obtain a value of zero. Therefore, Decoder-I performs the following two steps and iterates until no more coefficients can

be recovered.

- 1) Find all measurements $y_i = 0, i \in \{1, 2, \dots, m\}$ and set $\mathfrak{N}(y_i)$ to zero, where $\mathfrak{N}(y_i)$ denotes the set of coefficients of \underline{x} that are adjacent to y_i in graph G . This is because the non-zero coefficients have real values. Thus, it is almost impossible that the addition of some non-zero coefficients results in exact zero.
- 2) Find the measurements with only one unknown neighbor and recover the value of that neighbor accordingly.

We analyze the recovery probability of coefficients in set s_j employing Decoder-I. It should be noted that Decoder-I is similar to the belief propagation iterative decoding of rateless codes over erasure channels [21] with the additional step that all the neighboring coefficients of a zero-value measurement are set to zero.

We can rephrase the Decoder-I steps for our analysis as following. At every decoding iteration, 0 or 1 messages are sent along edges from coefficients to measurements and vice-versa. A measurement sends a 1 to a neighboring coefficient if and only if it is able to recover the coefficient's value. Similarly, a coefficient sends a 1 to its neighboring measurements if and only if its value has been recovered. In other words, a coefficient sends a 1 to a neighboring measurement if and only if it has received at least one message with value 1 from its other neighboring measurements. Therefore, we can say that the coefficients indeed do the logical OR operation. Moreover, a measurement sends 0 to a neighboring coefficient if and only if (i) the measurement is a non-zero measurement (this is imposed as a result of the first step of decoding in Decoder-I) and (ii) it has received at least one message with value 0 from its other neighboring coefficients. Accordingly, we infer that measurements do a combination of logical AND and OR operations.

Hence, we can extend the And-Or tree analysis technique, which has been used to analyze UEP rateless codes [7], to fit our problem and find the probability that a coefficient in s_j is not recovered (its value evaluates to 0) after l decoding iterations. Let us first obtain the decoding probability of the coefficients for uniform CS, i.e., $p_i = \frac{1}{n}, \forall i$. Similar to [7, 8], we choose a tree T_l , a subgraph of G , as follows. We choose an edge (v, w) uniformly at random from

¹The decoders in [18, 24] have additional steps compared to Decoder-I. However, we employ Decoder-I for making the analytical study tractable.

all edges in G . Call the coefficient v connected to edge (v, w) the root (depth 0). The subgraph of G induced by v and all neighbors of v within distance $2l$ after removing the edge (v, w) is shown to be a tree asymptotically [25]. In T_l , the children of v are at depth 1, their children at depth 2, and so forth. Nodes at depths $0, 2, 4, \dots, 2l$ are coefficients and the nodes at depths $1, 3, 5, \dots, 2l - 1$ are measurements. Performing l iterations of Decoder-I corresponds to considering the logical operations by the nodes (as explained above) in the tree T_l . If the root evaluates to logical 0 it means that the corresponding coefficient has not been recovered and vice versa. Let z_l be the probability that the root of T_l evaluates to 0 (is not recovered) after l decoding iterations for asymptotic case ($n \rightarrow \infty$). The following lemma formulates z_l .

Lemma 2: Assume a k -sparse signal \underline{x} of length n is encoded with a uniform CS, whose measurement matrix Φ has L non-zero entries uniformly distributed over each of its rows. Let z_l be the probability that a coefficient in \underline{x} is not recovered after l decoding iterations employing Decoder-I. We have $z_0 = 1$ and

$$z_l = \delta \left(1 - \beta(1 - z_{l-1}) - (1 - s) \left[\beta(1 - s) - \beta((1 - s)(1 - z_{l-1})) \right] \right), \quad l \geq 1, \quad (4)$$

where $\beta(x) = x^{L-1}$, $\delta(x) = e^{L\frac{m}{n}(x-1)}$, and $s = \frac{k}{n}$.

Proof: Proof is straightforward and similar to [7, Lemma 3]. We only emphasize that a measurement at depth $i + 1$ sends a 1 to a coefficient at depth i if either all its OR-node children at depth $i + 2$ send it a value 1 (second step of decoding in Decoder-I) or the measurement has a zero value (first step of decoding in Decoder-I). Considering the probability of these two events and subtraction of the probability of their intersection a measurement sends a 1 to a coefficient with probability $\beta(1 - z_{l-1}) + (1 - s)\beta(1 - s) - (1 - s)\beta((1 - s)(1 - z_{l-1}))$. Further, $\delta(x)$ can be easily obtained considering that the degree of coefficients (asymptotically) follows Poisson distributed with mean $L\frac{m}{n}$. ■

Next, we extend Lemma 2 to NCS with r importance levels.

Lemma 3: Assume a k -sparse signal of length n is encoded with a non-uniform CS with parameters $L, m, n, \underline{\alpha}$, and \underline{p} . Let $z_{l,j}$ be the probability that

a coefficient in s_j is not recovered after l decoding iterations employing Decoder-I. We have

$$z_{l,j} = \delta_j \left(1 - \beta \left(1 - \sum_{i=1}^r p_i \alpha_i n z_{l-1,i} \right) - (1 - s) \left[\beta(1 - s) - \beta((1 - s) \times (1 - \sum_{i=1}^r p_i \alpha_i n z_{l-1,i})) \right] \right), \quad l \geq 1, \quad (5)$$

where $z_{0,j} = 1$, $\beta(x) = x^{L-1}$, and $\delta_j(x) = e^{p_j L m (x-1)}$.

Proof: The proof is similar to Lemma 2 and [7, Lemma 3] and considering that the degree of coefficients in s_j follows Poisson distribution with mean $p_j L m$. ■

Definition 1: Define non-uniform gain $G_{l,i,j} \triangleq \frac{y_{l,i}}{y_{l,j}}$. This parameter compares the recovery probabilities of coefficients in s_i and s_j . A larger $G_{l,i,j}$ shows a higher decoding probability of coefficients in s_j in comparison with the coefficients in s_i .

It can be shown that

$$G_{l,i,j} = \exp \left[(p_j - p_i) m L \left(\beta \left(1 - \sum_{i=1}^r p_i \alpha_i n z_{l-1,i} \right) - (1 - s) \left[\beta(1 - s) - \beta((1 - s) \times (1 - \sum_{i=1}^r p_i \alpha_i n z_{l-1,i})) \right] \right) \right], \quad l \geq 1.$$

We can see that for $l \geq 1$, $G_{l,i,j} > 1$ if and only if $p_j > p_i$. Therefore, we can provide our desired non-uniform CS recovery by carefully tuning $p_j, j \in \{1, 2, \dots, r\}$ and setting a higher p_j for s_j 's with more importance levels.

A. Special Case with $r = 2$

In this section, a special case of NCS with Decoder-I and parameters $r = 2, L, n, m, \alpha_1 = \alpha, \alpha_2 = 1 - \alpha, p_1 = \frac{kM}{n}$, and $p_2 = 1 - p_1$ is investigated analytically. Let $z_{l,M}$ and $z_{l,L}$ denote the recovery probabilities of MICs and LICs at l^{th} decoding iteration employing Decoder-I, respectively. From Lemma 3, we have $z_{0,M} = z_{0,L} = 1$ and for $l \geq 1$

$$z_{l,M} = \exp \left(- \frac{k_M L m}{n} \beta \left(1 - (1 - \alpha) k_L z_{l-1,L} - \alpha k_M z_{l-1,M} \right) - (1 - s) \left[\beta(1 - s) - \beta((1 - s) \times (1 - (1 - \alpha) k_L z_{l-1,L} - \alpha k_M z_{l-1,M})) \right] \right), \quad (6)$$

$$z_{l,L} = \exp \left(- \frac{k_L L m}{n} \beta \left(1 - (1 - \alpha) k_L z_{l-1,L} - \alpha k_M z_{l-1,M} \right) - (1 - s) \left[\beta(1 - s) - \beta((1 - s) \times (1 - (1 - \alpha) k_L z_{l-1,L} - \alpha k_M z_{l-1,M})) \right] \right), \quad (7)$$

with $\beta(x) = x^{L-1}$.

In Figure 8, we evaluated the analytical results given by (6) and (7) for the probabilities that MICs and LICs are not recovered versus m . We have also depicted the probabilities that LICs and MICs are not recovered after l decoding iterations employing a real implementation of Decoder-I. In our simulations we set $n = 10^4$, $k = 100$, $L = 100$, $\alpha = 0.15$, and $k_M = 3$. Our asymptotic results in Figure 8 show that MICs are recovered with a much higher probability compared to LICs and confirms that desired non-uniform recovery can be provided employing NCS.

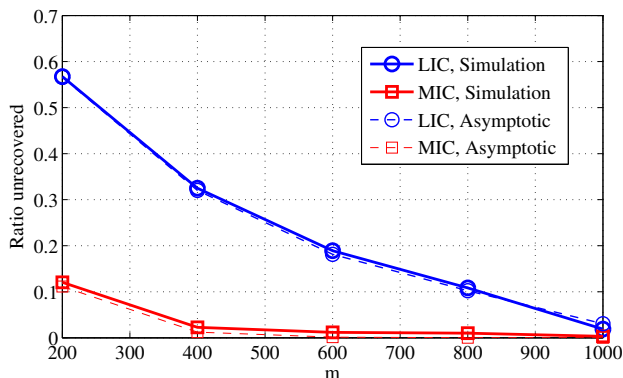


Fig. 8. The probability that coefficients in MICs and LICs are not recovered after l iterations of Decoder-I.

V. CONCLUSION

In this paper, we introduced *non-uniform compressive sensing* (NCS), which enables non-uniform recovery of the coefficients of sparse signals based

on their random projections. NCS is inspired by a previous work on *rateless codes* with *unequal-error-protection* property. A novel non-uniform sampling phase is designed such that the measurements capture the more important coefficients with a higher frequency. In this way, these coefficients are recovered with a higher precision compared to the less important coefficients. In NCS, only the encoding phase has been modified and the decoding remains intact. Therefore, any form of conventional CS reconstruction algorithm may be employed in the decoding phase of NCS. We showed that by employing NCS the desired higher reconstruction accuracy for more important parts of the signal can be obtained while the degradation in the reconstruction quality of the less important parts of data remains insignificant.

To make the theoretical analysis of NCS tractable, we also studied NCS under a simple (and suboptimal) decoding that is partly similar to the iterative decoding of rateless codes and analyzed NCS performance asymptotically by employing an extended And-Or tree analysis technique. Analytical results confirmed that the desired non-uniform recovery can be achieved by properly tuning NCS parameters.

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