# Generalized Unequal Error Protection Rateless Codes for Distributed Wireless Relay Networks

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***Abstract*—A generalization of unequal error protection (UEP) rateless codes and distributed rateless codes for distributed relay networks is proposed. We consider a two-hop relaying network where a single source transmits UEP rateless coded data to a destination via multiple relays. At the relays, the UEP rateless coded symbols are re-encoded by distributed rateless codes (DRC) to minimize the redundancy at the second hop. Previously introduced UEP rateless codes (URC) have supported a limited number of importance classes, mostly two classes, and targeted importance levels suited to a specific application. In this paper, however, we formulate an optimization problem to provide optimal URC in terms of the symbol error rate (SER) given any specific number and strengths of importance levels. Next, another optimization is proposed to obtain DRC with the minimum SER at a given overhead, which utilizes information of common symbols among the relays. The optimization methods are based on the AND-OR tree analysis and sequential quadratic programming. In addition, we evaluate the minimum achievable end-to-end symbol error rates over the wireless relay networks.**

***Index Terms*—Distributed rateless codes, unequal error protection, degree distribution optimization.**

## I. Introduction

RATELESS codes such as LT codes [1] and Raptor codes [2] are capacity-achieving loss-resilient codes for binary erasure channels. Rateless codes can recover all message symbols using a belief propagation (BP) decoder when a sufficient number of output symbols are received. A carefully designed degree distribution is used to recover $K$ message symbols with slightly more than $K$ output symbols. For example, the robust soliton distribution (RSD) [1] is an optimal degree distribution for LT codes, which allows them to recover message symbols with the minimum number of output symbols. Raptor codes [2] are concatenated codes with outer high-rate pre-codes and inner LT codes, which were designed to achieve linear encoding/decoding complexities. Note that LT and Raptor codes are basically designed for point-to-point and point-to-multipoint transmissions of equally protected data.

However, there are several systems where the classical rateless codes such as LT and Raptor codes cannot perform optimally. One example is the unequal error protection (UEP) system. In many applications, a portion of data needs to be protected with a higher reliability. Multimedia streaming services or wireless sensor networks (WSNs) are examples of such applications. For example, in MPEG video streaming services, I-frames need more protection than P-frames or B-frames. A WSN is also an application that can require UEP property. In a hostile environment, a faster recovery time is necessary for the data that contain crucial information regarding the environment. Unequal recovery time can be guaranteed by equipping the WSN with the UEP property. Another system is a non-point-to-multipoint or non-single-hop network. Recently, rateless codes have been considered in increasingly complicated networks that contain multiple sources or multiple relays. For example, in a large WSN, the data from distant sensors need to be collected at the base station through multi-hop relaying. In this case, we need efficient distributed rateless codes (DRC) capable of multi-source or multi-relay transmissions. Consequently, we need a method to provide UEP rateless coding (URC) schemes that perform optimally in more complicated networks.

Rateless codes that provide UEP property have been introduced to protect message symbols of which importance levels are differentiated. Rahnavard et al. [3] proposed the first rateless codes with UEP property. In [3], message symbols are selected with a non-uniform priority that corresponds to the importance level of each class of message symbols. Expanding window fountain codes [4] generate output symbols only from message symbols within a certain window. Overlapping and expanding windows are pre-designed such that any larger window contains all the symbols in the smaller windows. The representative URC in [3] and [4] have motivated several subsequent studies that consider many applications requiring UEP property. However, the authors of [3] and [4] introduced optimal ensembles only for the case of two importance levels. Hence, their proposed methods may not be able to provide optimal solutions for UEP applications requiring multi-level protections such as scalable video transmissions or distributed sensor networks. There have been numerous works [5]–[9] that modified the parameters or encoding/decoding algorithms of existing URC for the various applications. The authors of [10] proposed an optimization method to obtain URC with guaranteed Quality

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of Service. However, the optimization is executed with a predetermined degree distribution that is originally introduced for [2] and the results might be not globally optimal. No generalized optimizing method for the URC has been proposed, which can provide degree distributions with an arbitrary number of importance classes and required strength of importance levels.

DRC have been studied for systems with multiple sources. Distributed LT codes [11] and the soliton-like rateless coding scheme [12] provided degree distributions that generate LT-like output symbols in a relay with network coding protocol. Rateless codes for more complicated network models were introduced in [13]. The authors of [13] considered a multi-source multi-relay network with a single destination. Talari et al. [18] introduced distributed rateless codes that can provide UEP property when the numbers of message symbols of two sources differ. DRC for a network without NC have been also studied [14]–[15]. Shirvanimoghaddam et al. [14] demonstrated that the required minimum overhead can be reduced by appropriate cooperation of sources. Sejdinovic et al. introduced the decentralized distributed fountain coding scheme [15] that considers generalized rateless codes with noncooperative sources. AND-OR tree analysis was presented and some solvable code design problems based on linear programming (LP) were demonstrated in [16]. A further generalization of [15] is proposed in [17] by formulating a new optimization scheme based on the AND-OR tree analysis. However, in [17], the number of symbols generated by multiple sources are assumed to be identical, regardless of the number of symbols possessed by each source.

In this paper, we extend the result of [17] by considering a two-hop network with a single source, multiple relays, and a single destination. At the first hop, a source broadcasts the symbols generated by URC to multiple relays. We assume that the channels between the source and the relays are lossy. Each relay receives the same data with some randomly lost symbols. Thus some of the symbols are received by two or more relays at the same time. The relays then encode the received symbols with DRC to reduce the loss at the second hop. In this paper, we call the proposed URC and DRC generalized URC (GURC) and generalized DRC (GDRC), respectively. AND-OR tree analysis is used to obtain asymptotic performances for each code. We employ the sequential quadratic programming (SQP) optimization algorithm to utilize the results of performance analysis in order to design GURC and GDRC, which are optimal codes in terms of the symbol error rates. Note that we relax the assumption in [17], which restrict the number of generated symbols to be identical among the relays. With this relaxation, we can reduce the received overhead for the destination to recover data with high probability.

The work in this paper contains contributions in the following respects. First, the proposed optimization methods enable us to obtain optimal and generalized URC in a sense that we can obtain URC with the minimum symbol error rate (SER) given an arbitrary number of importance classes or strengths of importance levels. In other words, we can provide optimal URC even for the systems where conventional UEP schemes cannot provide any solution. Specifically, we try to minimize the asymptotic SER for the least important symbols while the UEP property is guaranteed for more important symbols. Second, we can obtain optimal DRC with the minimum SER by using information of common symbols among the relays. We can obtain a theoretical limit of the performance at the second hop for an ideal case by obtaining GDRC with some assumptions. For an ideal case, it is assumed that buffer size is enough for the relays to contain all the received symbols and each relay is aware of which symbol is received at the other relays. On the other hand, in practice, each relay employs LT codes individually without any additional information due to the limitation of resources, i.e., time, storage spaces, etc. With GDRC, we can estimate the minimum achievable symbol error rates at the second hop and room for improvement by observing the performance gap between an ideal and practical case. In addition, we can evaluate the end-to-end symbol error rates for the GURC over the wireless relay networks. Note that we focus on the generalization of URC and DRC rather than improving a specific URC for a particular application. The goal here is to transfer the source's data with unequal importance to the destination efficiently.

This paper is organized as follows. In Section II, we define the system model considered in this work. In Section III, we analyze the delay and amount of transferred data over the system. In Section IV, we propose GURC and GDRC and analyze their performances. In Section V, code design problems are formulated. Further, performances of the proposed and conventional schemes are evaluated and compared in Section VI. Section VII summarizes and discusses the results of this paper.

## II. System Model

Consider the system model in Fig. 1. A source $S$ delivers its message symbols to a destination $D$ via two-hop relaying. Here, $R_i,\ i = 1, \ldots, M$ denotes the $i$th relay, where $M$ is the number of relays. At $S$, $K$ message symbols are encoded with GURC into source-output (SO) symbols and broadcast to the $M$ relays. Then each relay re-encodes its received SO symbols with GDRC into relay-output (RO) symbols and transmits them to the destination. At $D$, two-step BP decoder is used to recover $K$ message symbols. Let $\epsilon_i$ and $\xi_i$, $i = 1, 2, \ldots, M$ denote the erasure rates of the source-relay (S-R) and relay-destination (R-D) channels, respectively. We assume that the erasure rates of all the channels are known to $S$ and $R_i$'s. The overall process of data transmission with the notation for the number of symbols is depicted in Figs. 1 and 2. Let $\gamma_{\mathrm{gurc}}$ and $\gamma_{\mathrm{gdrc}}$ denote a sufficient overhead to decode GURC and GDRC, respectively. Then the transmission of data along the network is executed as follows:

- Phase 1: A source $S$ generates and broadcasts $N_S = \gamma_S K$ SO symbols. Then $R_i$ receives $N_i = \gamma_S K(1 - \epsilon_i)$ SO symbols on average.
- Phase 2: After receiving $N_i$ SO symbols,
  - o Exchanging phase (included only when GDRC are used): Every $R_i$ reports the indices of the lost SO symbols to $S$ and then $S$ broadcasts the exact graph shown in Fig. 3(b) with the optimization results.
  - o Then $R_i$ generates $\frac{\gamma_R N_i}{1-\xi_i}$ RO symbols and transmits them to $D$.
- Phase 3: The destination $D$ utilizes two-step BP decoder to decode the received RO symbols into $K$ message symbols. The number of received RO symbols is $\gamma_R \sum_{i=1}^{M} N_i$ on average.

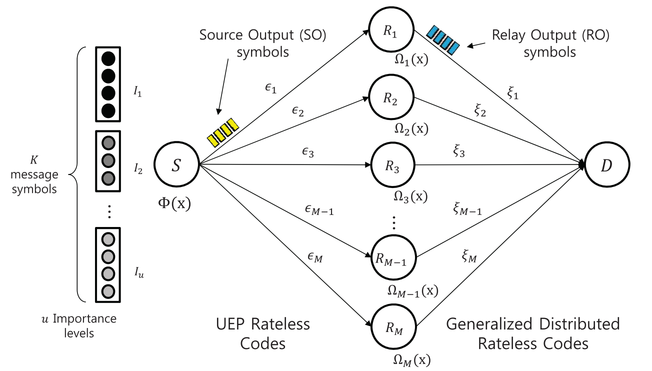



Fig. 1. Rateless coded data transmissions over a distributed relay network with a single source, $M$ relays, and a single destination.

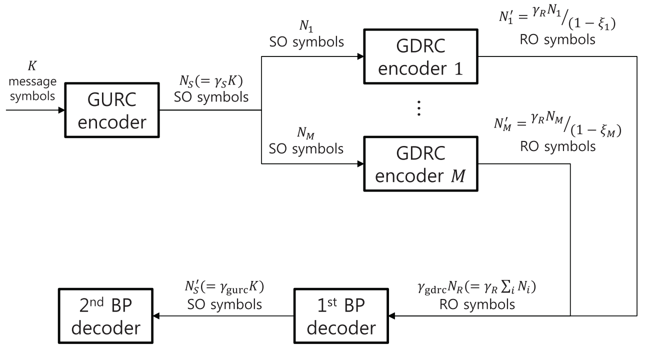



Fig. 2. Block diagram for the encoding and decoding process with notation for the number of symbols at each step.

One example of the wireless relay networks introduced above is a multi-user cooperation system which is similar to the system model considered in [14]. If a new user ($D$) comes into a network during broadcast phase (Phase 1), multiple existing users ($R_i$) can cooperate to transmit the data which they have been receiving from $S$ to the new user (Phase 2). In this case, it is assumed that $R_i$ has a buffer to store $N_i$ SO symbols before Phase 2. Note that each $R_i$ has a sufficient buffer to receive all of transmitted SO symbols and generates $\frac{\gamma_R N_i}{1-\xi_i}$ RO symbols so that $D$ receives $\gamma_R N_i$ RO symbols from $R_i$ on average. With this process, the number of the received RO symbols at $D$ becomes $\gamma_R \sum_{i=1}^{M} N_i = \gamma_{\text{gdrc}}\gamma_{\text{gurc}} K$. Next, specific structures of GURC and GDRC are introduced based on this setup.

For GURC, it is assumed that we have $u$ importance levels for our message symbols. The importance levels are represented by importance parameters $I_j$, $j = 1, \ldots, u$ which are normalized values proportional to the corresponding importance levels. For example, if the importance level of the 1st class is twice as much as that of the 2nd class, then $I_1$ and $I_2$ are determined such that $I_1 = 2I_2$ and $\sum_{j=1}^{u} I_j = 1$. Let $A_j$, $j = 1, \ldots, u$ denote a class of message symbols with the importance parameter $I_j$. The number of message symbols in each class $A_j$ is $\pi_j K$ where $\pi_j$, $j = 1, \ldots, u$ denotes the fraction of the message symbols corresponding to the $j$th (importance) class and $\sum_{j=1}^{u} \pi_j = 1$. Basically, in this paper, GURC employ the weighting method introduced in [3] and a degree distribution $\Phi(x) = \sum_d \Phi_d x^d$. Its corresponding bipartite graph is shown in Fig. 3(a). We note that the source selects a class $A_j$ with probability $p_j$, which is called the symbol-selection weight, with $\sum_{j=1}^{u} p_j = 1$. Given a degree $d$ chosen with probability $\Phi_d$, $S$ selects $d$ message symbols with probability $\frac{p_j}{\pi_j K}$ for the symbol in $A_j$. The selected message symbols are simply XORed and transmitted as in classical rateless codes. The number of SO symbols represented in Fig. 3(a) is $N'_S = \gamma_{\text{urc}} K$ since it is regarded as a decoding graph.

Let us consider the phase 2. Because of the random erasures, there exist SO symbols that are common between more than two relays. After every $R_i$ transmits all of their RO symbols, $D$ will see the received RO symbols with a decoding graph shown in Fig. 3(b) where $N_R = N_S(1 - \prod_{i=1}^{M} \epsilon_i)$ is the total number of received SO symbols without any repeated count of common symbols. Let $B_m$, $m = 1, \ldots, v$ denote disjoint sets of the symbols that are partitioned by the random erasures. The size of $B_m$ is $\rho_m N_R$ which is determined by channel erasure rates $\epsilon_i$'s. We have at most $v = 2^M - 1$ disjoint sets of received SO symbols. Although no relay has access to all $B_m$, we consider a graph shown in Fig. 3(b) with all $B_m$'s connected with all $R_i$'s for simplicity of graph reduction and further analysis. Similarly to GURC, each $R_i$ generate RO symbols with degree distribution $\Omega_i(x)$ and symbol-selection weights $w_{i,m}$. In practice, each $R_i$ simply employs LT codes with RSD and $w_{i,m}$ such that SO symbols are uniformly selected. Thus *Exchanging phase* is not necessary in this case. On the other hand, if GDRC are used, *Exchanging phase* is required so that all of $R_i$'s are informed of $\{\Omega_i(x), w_{i,m}\}$ which are optimal in terms of the average SER. It is assumed that all of the values of probabilities in this paper can be represented with integers of $q$ digits. Let $t = 10^q - 1$ be the maximum integer value with $q$ digits. Then the required complexity for *Exchanging phase* is at most $M \cdot N_S + M(d_{\max} + 2^M - 1) \cdot \lceil \log_2 t \rceil + M \cdot N_R(2^M - 1)$ bits, where $M \cdot N_S$ and $M(d_{\max} + 2^M - 1) \cdot \lceil \log_2 t \rceil + M \cdot N_R(2^M - 1)$ are the number of bits required to report the lost symbols and broadcast optimal $\{\Omega_i(x), w_{i,m}\}$ with the information of disjoint sets of SO symbols, respectively. It is shown that the complexity grows rapidly with $M$. Although it seems not practical to include *Exchanging phase* due to the high complexity, we investigate GDRC assuming the data transmission with *Exchanging phase*, in order to evaluate the minimum achievable SER. In this paper, the network where *Exchanging phase* is available is called an *ideal network*.

The destination employs the BP decoding algorithm to recover the message symbols sent from the source. For our system, a two-step BP decoding is executed because the received symbols are encoded twice by two different rateless codes. The generated RO symbols from each relay are considered to be symbols generated from a universal rateless code. The destination then peels off the graph shown in Fig. 3(b). to recover as many symbols as possible. The second decoding starts with the recovered SO symbols from the GDRC decoding. The message symbols are recovered by peeling off the graph shown in Fig. 3(a).

## III. Analysis of Delay and Sufficient Amount of Transferred Data

In order to verify an advantage of distributed coding at the relays, we analyze delay and the amount of transferred data based on the system model introduced in Section II. In Pakzad et al [19], a notion of delay is introduced and investigated for

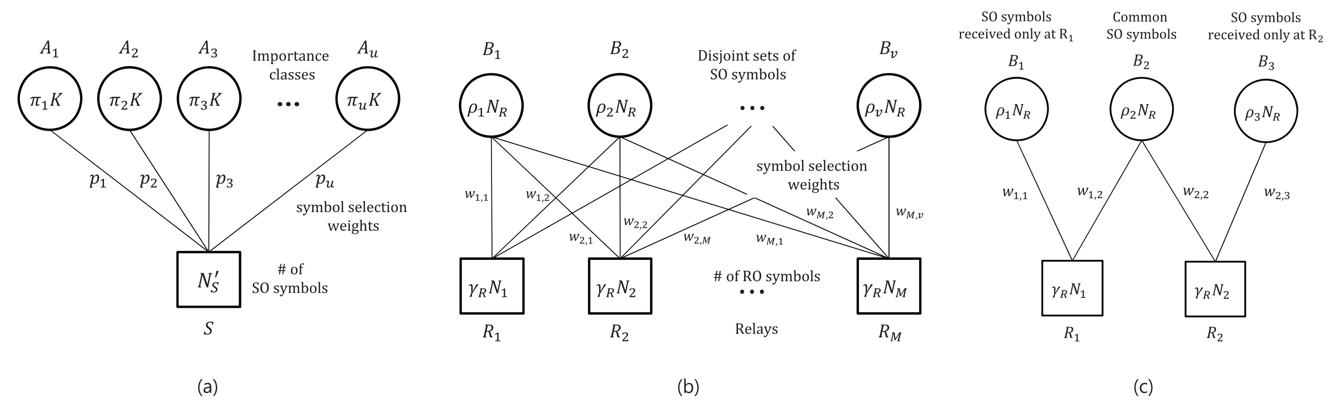



Fig. 3. (a) Generalized UEP rateless coding scheme with $N'_S$ SO symbols, $u$ importance levels, and the corresponding symbol-selection weights $p_j$ $j = 1, \ldots, u$. (b) Generalized distributed rateless coding scheme with $v$ disjoint message sets and $M$ relays. (c) Reduced bipartite graph for GDRC where $M = 2$.

various coding schemes on the line networks. It is assumed that a line network includes a source node $S$, an intermediate node $R$, and a destination node $D$, where the erasure rates of the first and the second channel are $\epsilon$ and $\xi$, respectively. Through the line networks (multiple nodes are serially arranged with erasure channels), the delay $L$ is defined in [19] as follows:

$$L = \Delta - \frac{K}{C_{\mathrm{mc}}} \tag{1}$$

where $K$ is the number of message symbols to be transferred to the destination, $\Delta$ denotes a duration of the end-to-end transmission of data for a specific scheme, and $C_{\mathrm{mc}}$ is the min-cut capacity. Thus $K/C_{\mathrm{mc}}$ is the ideal time of communication over a single channel with equivalent min-cut capacity. Let $n_1$ and $n_2$ be the number of symbols transmitted from $S$ and $R$, respectively. Then in order for $D$ to receive $n_2(1-\xi)$ symbols, $\Delta$ is required such that $\max\{n_1, n_2\} \le \Delta \le n_1 + n_2$. Furthermore, the transmission delay along channels is ignored as it is beyond our control. Based on this model, the delays of simple forwarding and the proposed scheme (GDRC) incurred by the relays are compared in our system model. For the simple forwarding scheme, a source utilize GURC and $M$ relays $R_i, i = 1, .., M$ just forward their received source output (SO) symbols to the destination $D$. In addition, since we discuss different schemes in the same system model, we can ignore the term $K/C_{\mathrm{mc}}$ in (1) when comparing the two schemes.

### A. Delay Incurred by the Relays

In the simple forwarding scheme, we can regard this network as $M$ parallel S-D channels with effective erasure rates $\psi_i$ such that $1 - \psi_i = (1 - \epsilon_i)(1 - \xi_i)$. Therefore, the delay is determined by the amount of transmitted SO symbols from $S$, such that $D$ receives $\gamma_{\mathrm{gurc}} K$ SO symbols. We have the communication time of simple forwarding scheme $\Delta_{\mathrm{SF}}$ as

$$\Delta_{\mathrm{SF}} = \frac{\gamma_{\mathrm{gurc}} K}{1 - \prod_{i=1}^{M} \psi_i}. \tag{2}$$

It seems that the simple forwarding is a quite efficient method in terms of delay since $\Delta_{\mathrm{SF}}$ approaches to $\gamma_{\mathrm{gurc}} K$ as $M$ increases. The impact of channel erasure rates reduces as the number of relays increases. Furthermore, $\Delta_{\mathrm{SF}}$ can be reduced by improving the performance of URC such that we have less $\gamma_{\mathrm{gurc}}$ with high probability.

In the proposed scheme, $M$ relays wait until $N_i = \gamma_{gurc} K(1 - \epsilon_i)/(1 - \prod_{i=1}^{M} \epsilon_i)$ SO symbols are received. Then $R_i$'s start transmitting $\frac{\gamma_R N_i}{1-\xi_i}$ RO symbols, where $\gamma_R = \gamma_{\mathrm{gurc}} \gamma_{\mathrm{gdrc}} K / \sum_{i=1}^{M} N_i$. Thus the duration is determined by the amount of transmitted SO symbols and the maximum number of transmitted RO symbols among those from $R_i$'s. We have a duration for the distributed coding scheme $\Delta_{\mathrm{DC}}$ as

$$\Delta_{\mathrm{DC}} = \frac{\gamma_{\mathrm{gurc}} K}{1 - \prod_{i=1}^{M} \epsilon_i} + \max_i \left\{ \frac{\gamma_{\mathrm{gurc}} \gamma_{\mathrm{gdrc}} K N_i}{(1 - \xi_i) \sum_{i=1}^{M} N_i} \right\}. \tag{3}$$

The duration $\Delta_{\mathrm{DC}}$ is longer than $\Delta_{\mathrm{SF}}$ due to the waiting time at the relays, which is a disadvantage of the distributed coding scheme. However, we can see that the gap between (2) and (3) is reduced when $M$ increases or $\gamma_{\mathrm{gdrc}}$ decreases as shown in Fig. 4. The first term in (3) is less than that of (2) in general as $\epsilon_i$ is less than $\psi_i$. It indicates that $S$ may transmit less amount of symbols in the distributed coding scheme than in the simple forwarding scheme. The second term can be reduced by improving the performance of GDRC (reducing $\gamma_{\mathrm{gdrc}}$) or increasing $M$. Therefore, if we have GURC and GDRC which are optimal in terms of decoding overhead, $\Delta_{\mathrm{DC}}$ can be minimized.

### B. Sufficient Amount of Symbols Transferred Along the Network

Although $\Delta_{\mathrm{DC}}$ is longer than $\Delta_{\mathrm{SF}}$, the proposed scheme is much more efficient method in terms of total amount of the transferred symbols along the network. Total number of symbols, $N_{\mathrm{SF}}$ and $N_{\mathrm{DC}}$, transmitted through S-R and R-D channels for the two schemes are easily obtained as follows:

$$N_{\mathrm{SF}} = \left( \frac{\gamma_{\mathrm{gurc}} K}{1 - \prod_{i=1}^{M} \epsilon_i} \right) \sum_{i=1}^{M} (1 - \epsilon_i) \tag{4}$$

$$+ \left( \frac{\gamma_{\mathrm{gurc}} K}{1 - \prod_{i=1}^{M} \psi_i} \right) \sum_{i=1}^{M} (1 - \psi_i).$$

$$N_{\mathrm{DC}} = \left( \frac{\gamma_{\mathrm{gurc}} K}{1 - \prod_{i=1}^{M} \psi_i} \right) \sum_{i=1}^{M} (1 - \epsilon_i) + \gamma_{\mathrm{gdrc}} \gamma_{\mathrm{gurc}} K. \tag{5}$$

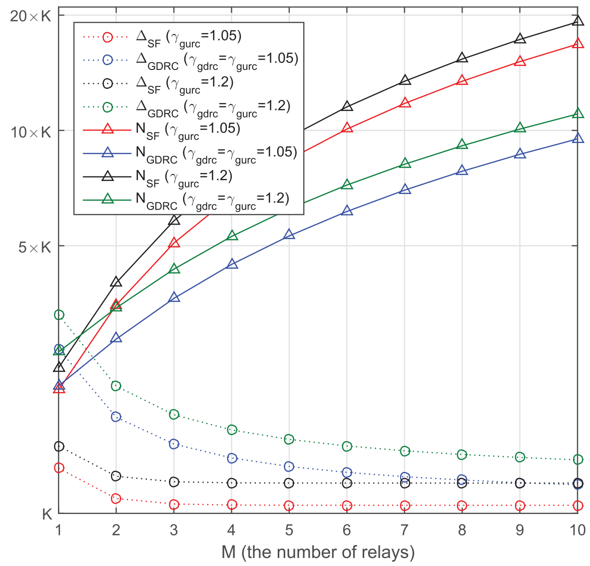



Fig. 4. Delay and sufficient number of symbols transferred along the network for simple forwarding (SF) scheme and distributed coding (DC) scheme.

For both (4) and (5), the first and second term stand for the amount of symbols received at the relays and the destination, respectively. The first term of (4) is less than that of (5) since $\psi_i$ is greater than $\epsilon_i$. It is also found in the second terms that we can reduce the amount of received symbols over R-D channels substantially using GDRC. The simple forwarding method incurs a large amount of redundant symbols since there are a lot of common SO symbols among the forwarded ones. On the other hand, using GDRC, each relay generate innovative and independent RO symbols. Thus $D$ can recover $K$ message symbols by receiving just $\gamma_{\text{gdrc}}\gamma_{\text{gurc}}K$ RO symbols. As shown in Fig. 4, gain of distributed coding in terms of the amount of the transferred symbols increases while the gap of delay between the two schemes decreases as $M$ increases or the decoding overhead, $\gamma_{\text{gurc}}$ or $\gamma_{\text{gdrc}}$, decreases. Next, we describe the asymptotic performances of GURC and GDRC, $\{\Phi(x), \{p_j\}_{j=1}^{u}\}$ and $\{\{\Omega_i(x)\}_{i=1}^{M}, \{w_{i,m}\}_{i=1,m=1}^{M,v}\}$, based on the AND-OR tree analysis technique.

## IV. AND-OR Tree Analysis of the Rateless Codes for Distributed Wireless Relay Networks

We theoretically estimate the decoding performances of the proposed rateless coding schemes prior to designing the codes for the considered system model. Since we assume that the BP algorithm is used for decoding in this paper, it is difficult to obtain the exact decoding failure probabilities without assuming an infinite length of message symbols. Therefore, for the sake of traceability of analysis, we assume that the number of message symbols at a source is infinitely large. To derive the asymptotic performance, AND-OR tree analysis [16] is employed. The recovery failure probability of a message symbol is obtained by AND-OR tree analysis. As introduced in [16], each probability at step $l$ is expressed as a recursive equation that is a function of the probability at the preceding step $l-1$.

For the double-hop relay network considered in this work, it is reasonable to analyze the performances of GURC and GDRC at once with all the information of the overall network. In other words, it is desirable to describe the recovery failure probability of a message symbol as a function of $\{\Phi(x), p_j, \Omega_i(x), w_{i,m}\}$, $i = 1, \ldots, M$, $j = 1, \ldots, u$, and $m = 1, \ldots, v$ by a single AND-OR tree analysis. In order to construct a single AND-OR tree, it should be possible to identify the degrees of RO symbols in terms of message symbols, which is the number of message symbols involved in generating each RO symbol. However, SO symbols are generated from the identical message symbol set, thus XOR operation of two disjoint SO symbols of degree $d_1$ and $d_2$ does not necessarily generates an RO symbol of degree $d_1 + d_2$. Therefore, it is not possible to interpret the two bipartite graphs, Figs. 3(a) and 3(b), in a single tree. Consequently, we have two separate AND-OR tree analyses for GURC and GDRC. For GURC at the S-R channel, we can obtain the results of the analysis for each class of the bipartite graph shown in Fig. 3(a). Similarly, for the GDRC at the R-D channels, the asymptotic error rates are defined for each disjoint set of the bipartite graph shown in Fig. 3(b).

### A. AND-OR Tree Analysis of the GDRC

We first formulate the AND-OR tree analysis for the GDRC used at the second hop so that we utilize the result to analyze the performance of GURC later in this section. Similar to [17], the AND-OR lemma for GDRC is given as follows.

*Lemma 1:* For all $m \in \{1, 2, \ldots, v\}$, the recovery failure probability $z_{m,l+1}$ of SO symbols from $B_m$ for an ensemble $(\{\Omega_i(x)\}_{i=1}^{M}, \{w_{i,m}\}_{i=1,m=1}^{M,v}, \{\rho_m\}_{m=1}^{v})$ at $(l+1)$-th iteration is given by

$$z_{m,l+1} = \exp\left[-\gamma_R \sum_{i=1}^{M} \frac{w_{i,m} N_i}{\rho_m N_R} \Omega_i'\left(1 - \sum_{h=1}^{v} w_{i,h} z_{h,l}\right)\right], \tag{6}$$

where $z_{m,0} = 1$ and $\gamma_R = \frac{N_i'(\xi_i)}{N_i}$ denotes the overhead defined as a ratio of the number of received RO symbols from each relay to the number of SO symbols.

*Proof:* Let $\mathcal{G}$ be the graph for GDRC, which is shown in Fig. 3(b). In $\mathcal{G}$, the received SO and RO symbols are mapped to input and output nodes, respectively. First, we focus on the subgraph $\mathcal{G}_i$, $i \in 1, 2, \ldots, M$ of the overall decoding graph $\mathcal{G}$ induced by output nodes only in $R_i$. The input nodes with non-zero degrees in $\mathcal{G}_i$ belong to the $B_m$ such that $w_{i,m} \neq 0$. In $\mathcal{G}_i$, the probability $\lambda_{d,m}^{i}$ that an input node in $B_m$ has degree $d$ is given by

$$\lambda_{d,m}^{i} = \binom{\mu_i \gamma_R N_i}{d} \left(\frac{w_{i,m}}{\rho_m N_R}\right)^d \left(1 - \frac{w_{i,m}}{\rho_m N_R}\right)^{\mu_i \gamma_R N_i - d}, \tag{7}$$

where $\mu_i = \Omega_i'(1)$ is the average output degree in $\mathcal{G}_i$. We can then approximate (7) by a Poisson distribution as

$$\frac{e^{-\alpha_m^i} (\alpha_m^i)^d}{d!}, \tag{8}$$

where $\alpha_m^i = \mu_i \gamma_R N_i \frac{w_{i,m}}{\rho_m N_R}$.

For the BP algorithm, input and output nodes execute the logical OR and AND operations, respectively. Thus we can map $\mathcal{G}_i$ to an AND-OR tree. We pick a disjoint set $B_m$ randomly and an arbitrary input node $a$ from the set. We consider an AND-OR tree for which $a$ is the root node. The OR-nodes in $B_m$ choose to have $d$ children in $R_i$ with probability $\delta^i_{d,m}$. It can easily be shown that

$$\delta^i_{d,m} = \frac{(d+1)\lambda^i_{d+1,m}}{\mu_i \gamma_R N_i \frac{w_{i,m}}{\rho_m N_R}} = \frac{e^{-\alpha^i_m}(\alpha^i_m)^d}{d!}. \tag{9}$$

In other words, this is the probability that an input node connected to a randomly selected edge has degree $d+1$ where the input node is in $B_m$. It can be represented in a polynomial form as

$$\begin{aligned}\delta^i_m(x) &= \sum_d \frac{e^{-\alpha^i_m}(\alpha^i_m)^d}{d!} \cdot x^d \\ &= e^{-\alpha^i_m} \sum_d \frac{(\alpha^i_m x)^d}{d!} = e^{\alpha^i_m (x-1)}.\end{aligned} \tag{10}$$

AND-nodes in $R_i$ choose to have $d$ children with probability $\beta_{i,d}$. This is the probability that a randomly selected edge in $\mathcal{G}_i$ is connected to an output node in $R_i$ of degree $d+1$, which can be shown as:

$$\beta_{i,d} = \frac{(d+1)\Omega_{i,d+1}}{\Omega'_i(1)}. \tag{11}$$

Hence,

$$\beta_i(x) = \frac{\Omega'_i(x)}{\Omega'_i(1)}. \tag{12}$$

We now consider the overall graph $\mathcal{G}$. Let the input node $a$ have degree $d_i$ within each subgraph $\mathcal{G}_i$ and $f^i_{m,l}$ be the probability that $a$ is not recovered at the $l$-th iteration. Then the conditional probability that $a$ is not recovered at the $(l+1)$-th iteration given degree $d_i$ can be shown as

$$\prod_{i=1}^{M} \{f^i_{m,l}\}^{d_i}. \tag{13}$$

Averaging over $d_i$, we can obtain the probability $z_{m,l+1}$ that $a$ is not recovered at the $(l+1)$-th as follows:

$$z_{m,l+1} = \prod_{i=1}^{M} \delta^i_m(f^i_{m,l}). \tag{14}$$

The input node $a$ can be recovered when at least one neighbor of $a$ can be released from the graph. Moreover, input nodes in $B_m$ becomes neighbors of an output node in $R_i$ independently with probability $w_{i,m}$. As a result, $f^i_{m,l}$ can be written as

$$f^i_{m,l} = 1 - \beta_i\left(1 - \sum_{h=1}^{v} w_{i,h} z_{h,l}\right) \tag{15}$$

Combining (10), (12), (14) and (15) results in the recovery failure probability of SO symbols from $B_m$. ■

The asymptotic performances $z_{m,l}$ for each disjoint message set $B_m$ are derived. Since there is correlation among the received symbols of the relays, the recovery of a symbol within a certain set can contribute to the recovery of another symbol within other sets. For example, we consider the case of two relays, $v = 3$, $M = 2$ as shown in Fig. 3(c). Let $S$ be assumed to be aware of the information of symbols shared among the relays, that is, the exact graph shown in Fig. 3(c). Then every $R_i$ can utilize GDRC and we can estimate the achievable performance of GDRC with specific symbol-selection weights as well as degree distributions. as stated in the Section II. On the other hand, if the information of common SO symbols cannot be informed to $S$ due to some practical issues, $R_1$ and $R_2$ have to employ LT codes individually with RSD using the symbol-selection weights $(w_{1,1}, w_{1,2}) = (\frac{\rho_1}{\rho_1+\rho_2}, \frac{\rho_2}{\rho_1+\rho_2})$ and $(w_{2,2}, w_{2,3}) = (\frac{\rho_2}{\rho_2+\rho_3}, \frac{\rho_3}{\rho_2+\rho_3})$, respectively. In this case, unlike GDRC, we cannot guarantee equal protection over the entire $N_R$ SO symbols since the SO symbols from $B_2$ excessively contribute to the RO symbols.

The SO symbols received by the $M$ relays should be equally protected. Therefore, based on the error rates of disjoint sets obtained by Lemma 1, we focus on the overall symbol errors which are the actual result of the decoding. Given the AND-OR tree analysis for GDRC, we can evaluate the asymptotic overall symbol error rates. According to Lemma 1, we have the following lemma.

*Lemma 2:* Let $z_{m,l}, m = 1, \ldots, v$ be the symbol error rates of input nodes in $B_m$ at the $l$-th iteration, then the overall symbol error rate $z_{\mathrm{gdrc},l}$ at the $l$-th iteration is

$$z_{\mathrm{gdrc},l} = \sum_{m=1}^{v} \rho_m z_{m,l} \tag{16}$$

*Proof:* Given the results in [16], [17], and [20], this result is straightforward. ■

From the decoder's point of view, the most important factor is the overhead since it significantly affects the decoding performance. In other words, the result of the first decoding is crucial to the result of the second decoding since the destination recovers message symbols with a two-step decoding. Therefore, $z_{\mathrm{gdrc},l}$ is an important factor that should be minimized.

### B. AND-OR Tree Analysis of the GURC

Next, we formulate the AND-OR tree analysis of GURC used at the first hop (S-R channel). Because the GURC decoder is employed after the GDRC decoding is finished, the number of symbols recovered by the GDRC decoder is a dominant factor in the performance of GURC. As stated in the Section II, we choose to use [3] in order to make the overall analysis simple and tractable. Although various subsequent studies have been introduced, they mostly kept following the principle of weighting [3] and proposed a specific encoding/decoding protocol without designing degree distributions. Therefore it is reasonable to employ the principle of weighting [3]. The AND-OR tree analysis of GURC is derived using the result of (6) as follows.

*Lemma 3:* For all $j \in \{1, 2, \ldots, u\}$, the recovery failure probability $y_{j,l+1}$ of message symbols from $j$-th class for an ensemble $(\Phi(x), \{p_j\}_{j=1}^{u}, \{\pi_j\}_{j=1}^{u})$ at $(l+1)$-th iteration is given by

$$y_{j,l+1} = \exp\left[-\gamma_{\mathrm{gurc}} \frac{p_j}{\pi_j} \Phi'(1 - \sum_{r=1}^{u} p_r y_{r,l})\right] \tag{17}$$

where $y_{j,0} = 1$ and $\gamma_{\mathrm{gurc}}$ denotes the decoding overhead defined as the ratio of the number of SO symbols recovered by the GDRC decoder to the number of message symbols.

*Proof:* The bipartite graph shown in Fig. 3(a) is basically the same as in [3]. Given the proof of Lemma 3 in [3], this result is straightforward. ■

Because there are some symbols erased over the S-D channel and not recovered from the GDRC decoder, the decoding overhead $\gamma_{\mathrm{gurc}}$ of the GURC decoder is determined by the encoding overhead $\gamma_S$, channel erasure rates $\epsilon_i, i = 1, \ldots, M$, and the result of the first decoding $z_{\mathrm{gdrc},l}$. Given a fraction $\pi_j$ and overhead $\gamma_{\mathrm{gurc}}$, symbol error rate of each class is a function of $\Phi(x) = \sum_d \Phi_d x^d$ and $p_j$.

## V. Design of Rateless Codes for Distributed Wireless Relay Networks

Several studies have attempted to optimize the degree distribution of their proposed rateless codes. LP based techniques have been formulated in several previous studies because they can find globally optimal solutions. The authors of [14] and [15] defined the objective function as an overhead with a fixed average input degree. The result of [15] allows us to obtain optimal rateless codes only for some special cases of GDRC. The conventional LP method regards all the decoding failure probabilities, $z_{m,l}, m = 1, 2, \ldots, v$, of each class as a function over a single variable, i.e., if $z_{1,l} = \lambda$, then $z_{2,l} = f(\lambda)$, $z_{3,l} = g(\lambda)$ and so on. This happens only when the SO symbols are symmetrically overlapped[1] and uniformly selected. SO symbols are referred to as being symmetrically overlapped when there is only one common SO symbol set $A_c, c \in \{1, 2, \ldots, v\}$ that is shared by all the other relays where the others have the same fraction $\rho_m$, $m \neq c$ with identical degree distributions $\Omega(x) = \Omega_1(x) = \cdots \Omega_M(x)$. However, since random erasures are distributed uniformly over the SO symbols, we are unlikely to obtain solutions using the method in [15].

For the network model considered in this work, we need to cope with all possible cases such as channel erasure rates, the number of relays, and the number of message importance levels. Moreover, in order to obtain the symbol-selection weights as well as the degree distributions, a new optimization scheme is required. As stated above, it is impossible to represent the overall performance as a function of $\{\Phi(x), p_j, \Omega_i(x), w_{i,m}\}, i = 1, \ldots, M, j = 1, \ldots, u$, and $m = 1, \ldots, v$ at once. Thus, GURC and GDRC must inevitably be designed by a two-step optimization. Note that, in our optimization, we focus on the water-fall region that is important for broadcasting communication scenarios. It is shown later in this paper that we can achieve SER of under $10^{-5}$ with the proposed scheme.

[1] Similarly to [17], the SO symbols that are used to generate the RO symbols by two or more relays are here said to be overlapped.

### A. Design of GDRC

For the first decoding, we focus on the recovery of the overall $N_R$ symbols rather than those of a certain disjoint set. Therefore, we need to define the overall symbol error rate by combining each error rate at every iteration. An objective function for the first decoding can be defined based on the AND-OR tree analysis and Lemma 2.

*Definition 1:* For the first decoding, the overall symbol error rate is determined as an objective function $R_{\mathrm{gdrc}}(\{\Omega_i(x)\}_{i=1}^{M}, \{w_{i,m}\}_{i=1,m=1}^{M,v})$, which is a function of degree distributions $\Omega_i(x), i = 1, \ldots, M$ and symbol-selection weights $w_{i,m}, i = 1, \ldots, M, m = 1, \ldots, u$.

$$R_{\mathrm{gdrc}}(\{\Omega_i(x)\}_{i=1}^{M}, \{w_{i,m}\}_{i=1,m=1}^{M,v}) \triangleq \lim_{l \to \infty} z_{\mathrm{gdrc},l} \tag{18}$$

where $z_{\mathrm{gdrc},l}$ is the symbol error rate at the $l$-th iteration step, as shown in Lemma 2.

LT codes employed by the relays utilize fixed symbol-selection weights $\{w_{i,m}^{\mathrm{RSD}}\}_{i=1,m=1}^{M,v}$ and RSD degree distributions. However, the objective function for GDRC is defined as a function of symbol-selection weights as well as degree distributions since we consider an ideal network to estimate the best-effort performance. The way we evaluate the objective function and define the optimization problem is similar to [20]. The authors of [20] defined an optimization problem and solved it to obtain rateless codes with a minimum average symbol error rate for the dying binary erasure channels (DBEC), where DBEC is defined as a binary erasure channel being randomly terminated during transmissions. However, unlike [20], the objective function in this work is defined as the convergent value of the symbol error rate at a fixed overhead $\gamma_{\mathrm{gdrc}}$. Here, $R_{\mathrm{gdrc}}(\{\Omega_i(x)\}_{i=1}^{M}, \{w_{i,m}\}_{i=1,m=1}^{M,v})$ is a nonlinear function over the values of $\Omega_{i,d}$ and $w_{i,m}$ since the asymptotic performances are represented by exponential functions. We can define the optimization problem as follows.

$$\begin{aligned} \min_{\Omega_i(x), w_{i,m}} \; & R_{\mathrm{gdrc}}\left(\{\Omega_i(x)\}_{i=1}^{M}, \{w_{i,m}\}_{i=1,m=1}^{M,v}\right) \\ \text{s.t.} & \sum_{d=1}^{d_{\max}} \Omega_{i,d} = 1, \Omega_{i,d} \geq 0, \\ & 0 \leq w_{i,m} \leq 1 \\ & \Omega_i'(1) \geq \log(N_i), \\ & d = 1, \ldots, d_{\max}, i = 1, \ldots, M, m = 1, \ldots, v \end{aligned} \tag{19}$$

The goal here is to obtain degree distributions $\Omega_i(x), i = 1, 2, \ldots, M$ and symbol-selection weights $w_{i,m}, i = 1, 2, \ldots, M, m = 1, \ldots, v$ that minimize the symbol error rate at a certain overhead $\gamma_{\mathrm{gdrc}}$. In (19), $d_{\max}$ denotes the largest degree of all $\Omega_i(x)$. The constraints on lines 2 and 3 are the necessary conditions for the values of $\Omega_{i,d}$ and $w_{i,m}$ to be a valid probability. The constraint on line 4 stands for a lower bound of the average degree, $\Omega_i'(1)$, such that error floors in the waterfall region are

TABLE I
OPTIMAL SYMBOL-SELECTION WEIGHTS $w^*_{i,m}$, $i \in \{1, 2\}$, $m \in \{1, 2, 3\}$ AND DEGREE DISTRIBUTIONS $\Omega^*_i(x)$, $i \in \{1, 2\}$ FOR GDRC AT $\gamma_R = 1.05$ WITH $M = 2$

| Erasure rates | Optimal Weights | Optimal Degree Distributions |
|---|---|---|
| $\epsilon_1 = 0.15$<br>$\epsilon_2 = 0.2$ | $w^*_{1,1} = 0.2568$<br>$w^*_{1,2} = 0.7432$<br>$w^*_{2,2} = 0.3015$<br>$w^*_{2,3} = 0.6985$ | $\Omega^*_1(x) = 0.0797x + 0.3752x^2 + 0.1036x^3 + 0.161x^4 + 0.0342x^5 + 0.0181x^6$<br>$+0.0159x^7 + 0.0137x^8 + 0.0278x^9 + 0.0365x^{10} + 0.0423x^{11} + 0.0273x^{12}$<br>$+0.0074x^{13} + 0.0055x^{15} + 0.0195x^{99} + 0.0418x^{100}$<br>$\Omega^*_2(x) = 0.0054x + 0.4394x^2 + 0.2024x^3 + 0.2179x^4 + 0.0162x^8 + 0.0106x^9$<br>$+0.0175x^{10} + 0.0143x^{12} + 0.0111x^{22} + 0.0089x^{32} + 0.0096x^{67}$<br>$+0.0111x^{68} + 0.0356x^{100}$ |
| $\epsilon_1 = \epsilon_2 = 0.2$ | $w^*_{1,1} = w^*_{2,3}$<br>$= 0.2210$<br>$w^*_{1,2} = w^*_{2,2}$<br>$= 0.7790$ | $\Omega^*_1(x) = \Omega^*_2(x) = 0.0144x + 0.4613x^2 + 0.1513x^3 + 0.0767x^4 + 0.0466x^5$<br>$+0.0312x^6 + 0.0223x^7 + 0.0166x^8 + 0.015x^9 + 0.0151x^{10} + 0.0149x^{11}$<br>$+0.0143x^{12} + 0.0133x^{13} + 0.0119x^{14} + 0.01x^{15} + 0.0078x^{16} + 0.0085x^{17}$<br>$+0.0096x^{19} + 0.0099x^{22} + 0.0064x^{28} + 0.0208x^{187} + 0.0036x^{190}$<br>$+0.0052x^{193} + 0.0078x^{196} + 0.0032x^{199} + 0.0035x^{200}$ |

TABLE II
OPTIMAL SYMBOL-SELECTION WEIGHTS $p^*_j$, $j \in \{1, \ldots, u\}$ AND DEGREE DISTRIBUTION $\Phi^*(x)$ FOR GURC AT $\gamma_{\text{GURC}} = 1.05$ WITH $u = 2$ AND $u = 3$, RESPECTIVELY

| Importance levels | Optimal Weights | Optimal Degree Distribution |
|---|---|---|
| $I_1 = 10I_2$<br>$I_1 + I_2 = 1$ | $p^*_1 = 0.1789$<br>$p^*_2 = 0.8211$ | $\Phi^*(x) = 0.0099x + 0.4249x^2 + 0.2615x^3 + 0.0671x^5 + 0.0516x^7 + 0.0115x^8$<br>$+0.0181x^9 + 0.0296x^{11} + 0.0071x^{12} + 0.009x^{14} + 0.0171x^{15} + 0.0253x^{16}$<br>$+0.0088x^{17} + 0.0585x^{100}$ |
| $15I_1 = 100I_2$<br>$I_2 = 15I_3$<br>$I_1 + I_2 + I_3 = 1$ | $p^*_1 = 0.2795$<br>$p^*_2 = 0.2918$<br>$p^*_3 = 0.4287$ | $\Phi^*(x) = 0.0527x + 0.1482x^2 + 0.3889x^3 + 0.0206x^4 + 0.027x^5 + 0.0069x^6$<br>$+0.0411x^7 + 0.0231x^8 + 0.0673x^9 + 0.0215x^{10} + 0.0237x^{11} + 0.0267x^{12}$<br>$+0.0546x^{13} + 0.0064x^{16} + 0.0096x^{17} + 0.0071x^{18} + 0.0022x^{19} + 0.002x^{24}$<br>$+0.0019x^{97} + 0.0048x^{98} + 0.0085x^{99} + 0.0552x^{100}$ |

mitigated. Since $R_{\text{gdrc}}(\{\Omega_i(x)\}_{i=1}^M, \{w_{i,m}\}_{i=1,m=1}^{M,v})$ is a nonlinear function and is generally non-convex, the SQP algorithm is adopted to find a locally optimal solution similar to [20]. Given $\{\{\rho_m\}_{m=1}^v, \{N_i\}_{i=1}^M\}$, the optimal parameters $(\{w_{i,m}\}_{i=1,m=1}^{M,v}, \{\Omega_i(x)\}_{i=1}^M)$ are obtained. This method enables us to design optimal rateless codes for all possible correlations of symbols, $\{\rho_1, \rho_2, \ldots, \rho_v\}$. Some results of the optimization for GDRC are presented in Table I.

## B. Design of GURC

After the optimization of GDRC is completed, the degree distribution and the symbol-selection weights of GURC are optimized. Message symbols are separated into $u$ classes $A_j$, $j = 1, \ldots, u$ according to their importance levels $I_j$, $j = 1, \ldots, u$. Thus the optimization problem is defined so that the solution reflects the importance levels of the message symbols. The goal is to find $\Phi(x)$ and $p_j$, $j = 1, \ldots, u$ that minimize the symbol error rate for each importance class while providing UEP property. The optimization problem for GURC is formulated as follows.

$$\begin{aligned}
&\min_{\Phi(x), p_j} R_{\text{gurc}}\left(\Phi(x), \{p_j\}_{j=1}^u\right) \\
&\text{s.t.} \sum_{d=1}^{d_{\max}} \Phi_d = 1, \Phi_d \geq 0, \\
&0 \leq p_j \leq 1, \Phi'(1) \geq \log(K), \\
&I_1 y_1 \leq I_2 y_2 \leq \cdots \leq I_u y_u, \\
&d = 1, \ldots, d_{\max}, j = 1, \ldots, u
\end{aligned} \tag{20}$$

where $R_{\text{gurc}}(\Phi(x), \{p_j\}_{j=1}^u) = \lim_{l \to \infty} y_{u,l}$ denotes the convergent of asymptotic SER of the least important class.

Let the importance levels of message symbols be sorted as $I_1 > I_2 > \cdots > I_u$ without loss of generality, then the objective function can be written as $R_{\text{gurc}}(\Phi(x), \{p_j\}_{j=1}^u) = \lim_{l \to \infty} y_{u,l}$. Note that the inequalities on the fourth line of (20) are added to provide UEP property. Other constraints are the same as for the optimization of GDRC introduced in the previous subsection. Because of the inserted nonlinear inequalities, minimizing the symbol error rate of the least important symbol forces the symbol error rates of other classes to be minimized according to their importance level. The constraint on the average degree is added to reduce the error floor. The resulting degree distributions are shown in Table II and it is found that the average degrees $\Phi'(1)$ for $u = 2$ and $u = 3$ are 9.9763 and 11.85, respectively. Each of them has about 1.75 and 2.12 times as many edges as Raptor codes, $\Phi'_{\text{conv}}(1) = 5.5764$ on average, which incur the higher decoding complexity. However, it is shown that GURC show SEE about 10 times as low as [3] at the overhead $\gamma_{\text{gurc}} = 1.15$. In this case, $R_{\text{gurc}}(\Phi(x), \{p_j\}_{j=1}^u)$ is also a nonlinear function and generally non-convex. Therefore, the SQP algorithm is adopted in a similar way as for GDRC. Some results of the optimization for GURC are presented in Table II.

It is easily noticed that the final optimization is affected by the results of the previous decoding for GDRC. The decoding overhead for GURC, $\gamma_{\text{gurc}}$, is determined by the number of recovered symbols after the first decoding. Note that the SO symbols are lost over the erasure channels with $\epsilon_i$, $i = 1, \ldots, M$ and the BP decoder for GDRC. Therefore, it is straightforward to determine the decoding overhead for GURC

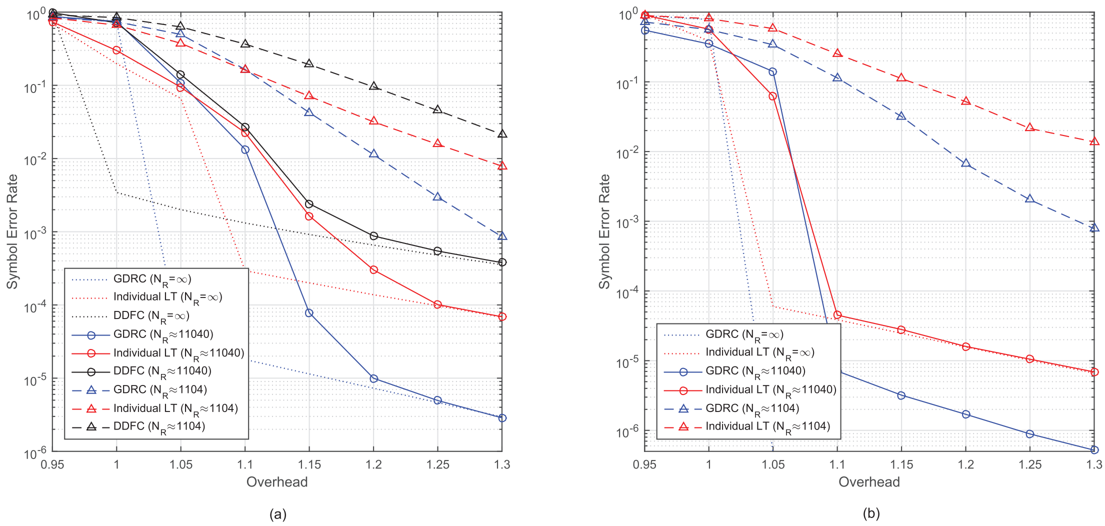



Fig. 5. (a) SERs of GDRC, the individual LT codes, and DDFC for the second hop, where $M = 2$ and $\epsilon_1 = \epsilon_2 = 0.2$. (b) SERs of GDRC and the individual LT codes for the second hop, where $M = 3$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.2$.

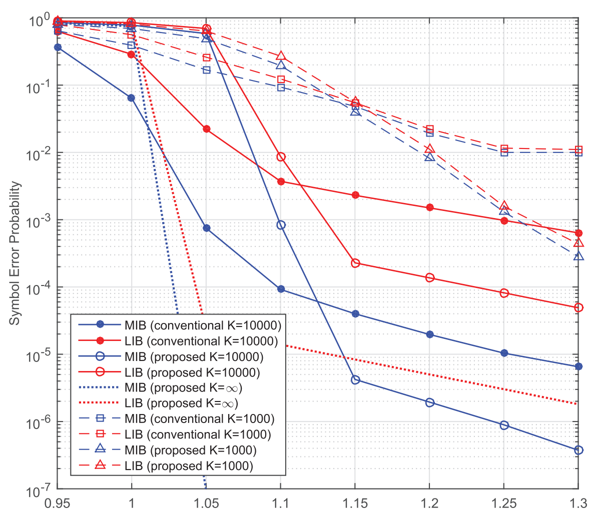


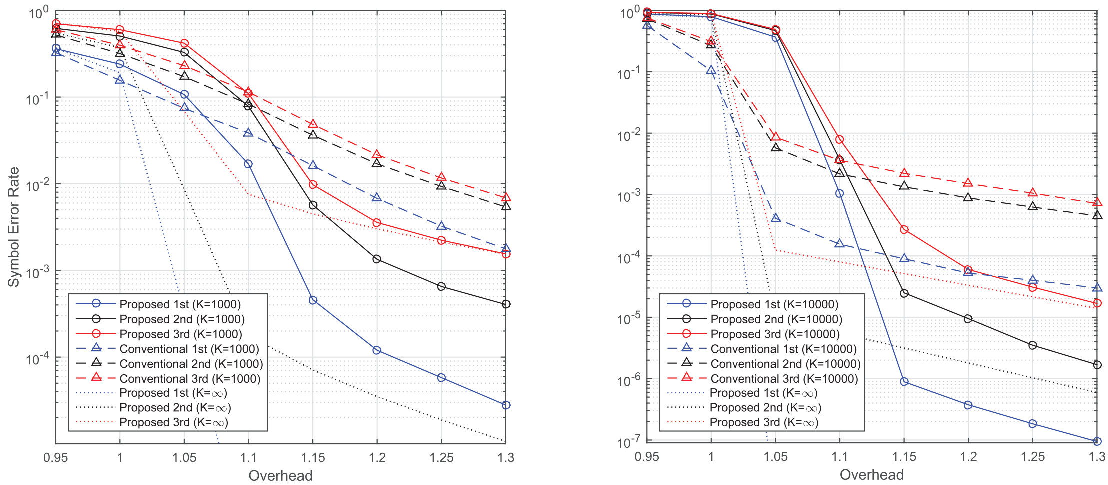

Fig. 6. SERs of the proposed GURC and the conventional URC for the first hop, where $u = 2$, $\pi_1 = 0.1$, $\pi_2 = 0.9$, and $I_1 = 10I_2$.

as $\gamma_{\text{gurc}} = N_R(1 - \lim_{l\to\infty} z^*_{\text{gdrc},l})$ where $z^*_{\text{gdrc},l}$ is the symbol error rate of GDRC with optimal degree distributions $\Omega^*_i(x)$ and symbol-selection weights $w^*_{i,m}$, $i = 1, \ldots, M$, $m = 1, \ldots, v$. Because a decoding overhead is the most important factor to ensure that BP decoders have waterfall regions, both of the optimizations should provide optimal ensembles for reliability. Experimental results show that the proposed method can provide better rateless codes than the previous UEP and distributed rateless codes for the proposed network model.

## VI. Experimental Results

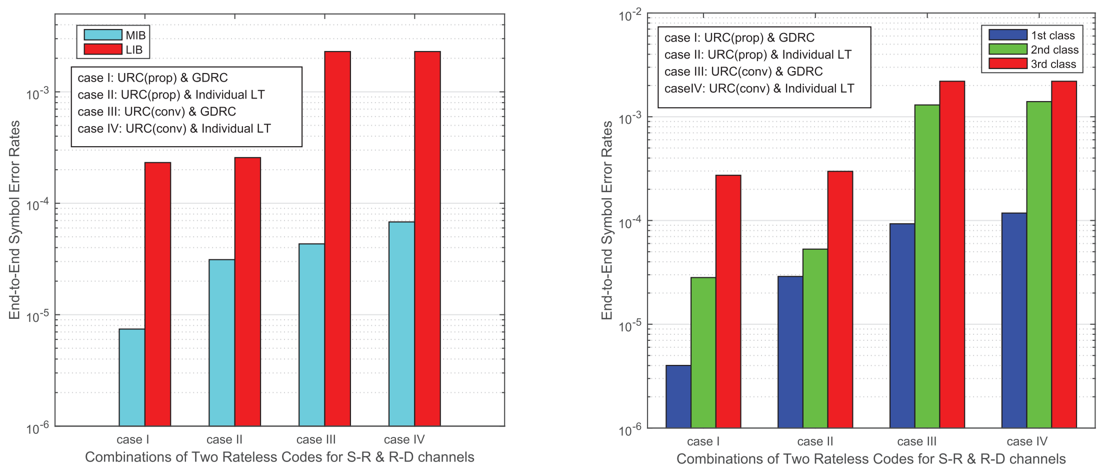
Optimal degree distributions and symbol-selection weights for GDRC and GURC are shown in Tables I and II, respectively. The symbol-selection weights for GDRC are determined such that equal error protection is guaranteed while those for GURC are determined such that they provide the corresponding UEP property. In the Table I, the optimal solutions for GDRC with $M = 2$ are shown. We note that the solutions are affected by the erasure rates $\epsilon_i$ since the fraction of distinct set of symbols are determined by the erasure rates. In the first row of Table I, it is found that $R_1$ is more likely to select symbols in $B_2$ than $R_2$. When $\epsilon_1 = \epsilon_2$ as shown in the second row, relays select SO symbols with the identical weights. In Table II, optimal solutions for GURC for $u = 2$ and $u = 3$ are shown. Note that Table I and II show the optimization results for $K = 10000$. We would have different results with different $K$ since we have constraints for $K$ in our optimization as shown in (19) and (21). The total number of message symbols used in this simulation was $K = 10000$ and $K = 1000$. With these optimization results, we evaluated performances of GDRC and GURC. End-to-end performances for the entire network were then compared.

First, we compared the performances of the GDRC, Individual LT codes, and decentralized distributed fountain codes (DDFC) [15] for the second hop transmission. GDRC shows the performance for an ideal network, in which the relays can construct the exact graph as Fig. 3(b). However, in practice, it is difficult to assume that the information about common symbols among relays is available, which includes the fraction of the disjoint sets $B_m$, $m = 1, \ldots, v$ and symbol indices within those sets. As the relays cannot construct the exact graph as Fig. 3(b), they simply utilize LT codes individually. Given the received SO symbols, $R_i$ utilizes RSD to encode $N_i$ SO symbols into RO symbols, where the SO symbols are uniformly selected during the encoding process. We employed an RSD with the parameters $c = 0.03$ and $\delta = 0.5$. The performance of the optimal GDRC is evaluated to estimate how much we can improve the performance against the individual LT codes. Code parameters shown in Table I are used for the optimal GDRC. As shown in [17], DDFC is available only for the case of $\epsilon_1 = \epsilon_2$ when $M = 2$. Following the LP optimization in [15], we have $\Omega_{\text{DDFC}}(x) = 0.002x + 0.4474x^2 + 0.2068x^3 + 0.1214x^5 + 0.0373x^6 + 0.0687x^{10} + 0.0196x^{11} + 0.0414x^{24} + 0.0172x^{25} + 0.0079x^{97}$




(a) (b)

Fig. 7. SERs of the proposed GURC and the conventional URC where $u = 3$, $\{\pi_1, \pi_2, \pi_3\} = \{0.1, 0.2, 0.7\}$, and $15I_1 = 100I_2$, $I_2 = 15I_3$ (a) for $K = 1000$ and (b) $K = 10000$.




(a) (b)

Fig. 8. (a) End-to-end symbol error rates for $u = 2$ and (b) end-to-end symbol error rates for $u = 3$, where $M = 3$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.2$, $u = 3$, $K = 10000$, $\gamma_{\text{gurc}} = 1.15$, and $\gamma_{\text{gdrc}} = 1.1$.

$+0.0102x^{98} + 0.0101x^{99} + 0.01x^{100}$. Given erasure rates $\epsilon_i, i = 1, \ldots, M$ of S-R channels, we evaluate symbol error rates (SER) of the three schemes, GDRC, individual LT codes, and DDFC. We plot the SER for the case of $M = 2$ and $\epsilon_1 = \epsilon_2 = 0.2$ for $M = 2$ and $M = 3$ as shown in Figs. 5(a) and 5(b), respectively. Note that DDFC cannot give any solution for $M = 3$ case, which has been studied in [17]. Thus we have performances of DDFC only for $M = 2$ case.

In this case, the BP decoder tries to recover $N_R = \gamma_{\text{gurc}} K$ SO symbols from $\gamma_{\text{gdrc}} N_R$ RO symbols. Since we assumed that $\gamma_{\text{gurc}} = 1.15$, the number of symbols to be recovered by the GDRC decoder for $M = 2$ case was $N_R \approx 11040$ and $N_R \approx 1104$ when $K = 10000$ and $K = 1000$, respectively. For $M = 3$ case, we have $N_R \approx 11,400$ and $N_R \approx 1140$ when $K = 10000$ and $K = 1000$, respectively. Asymptotic performance ($N_R = \infty$) shows us an expected performance gap between the proposed GDRC and the LT codes. For $M = 2$ and $N_R = 11040$, GDRC, individual LT codes, and DDFC result in a SER of $7.8 \times 10^{-5}$, $1.68 \times 10^{-3}$, and $2.41 \times 10^{-3}$, respectively, at the overhead $\gamma_{\text{gdrc}} = 1.15$ as shown in Fig. 5(a). GDRC shows much enhanced performance than other schemes. For the individual LT codes and DDFC, the symbols within set $B_2$ excessively participate in the encoding due to the uniform selection. The symbols in $B_2$ become redundant thus they can no longer contribute to decoding. For $M = 3$ and $N_R = 11040$, the optimal GDRC results in a SER of $7.1 \times 10^{-6}$, whereas the individual LT codes results in a SER of $4.5 \times 10^{-5}$, at $\gamma_{\text{gdrc}} = 1.1$ as shown in Fig 5(b). For both of the $M = 2$ and $M = 3$ cases, GDRC still shows the lowest SER for the short length messages even though it is degraded. As we expected in Section III and shown in Fig. 5(a) and 5(b), a sufficient overhead is reduced as $M$ increases.

Next, we evaluated the performance of the proposed GURC. Given the fraction $\{\pi_j\}_{j=1}^{u}$ of the message set for the corresponding importance level $I_j, j = 1, \ldots, u$, the SERs of the proposed GURC were compared with the conventional

URC [3]. Since in [3] the symbol-selection weight for $u = 2$ has been optimized given the degree distribution $\Phi_{\mathrm{conv}}(x) = 0.0080x + 0.4936x^2 + 0.1662x^3 + 0.0726x^4 + 0.0826x^5 + 0.0561x^8 + 0.0372x^9 + 0.0556x^{19} + 0.0250x^{65} + 0.0031x^{66}$, we use $\Phi_{\mathrm{conv}}(x)$ to compare the performance. On the other hands, the proposed GURC employs $\Phi^*(x)$ and the values for $p_j^*$ as shown in Table II. In case of $u = 2$, SERs for each class are shown in Fig. 6 where the fractions of the MIB and LIB classes are $\pi_1 = 0.1$ and $\pi_2 = 0.9$, respectively, and the corresponding importance levels are $I_1 = \frac{10}{11}$ and $I_2 = \frac{1}{11}$, respectively. The symbol-selection weights used for the conventional scheme were $p_1 = 0.2077$, $p_2 = 0.7923$, as used in [3]. For the MIB class, the proposed GURC results in SER of $4.23 \times 10^{-6}$ while the conventional URC results in SER of $4.02 \times 10^{-5}$ at the overhead $\gamma_{\mathrm{gurc}} = 1.15$ when $K = 10000$. For the LIB class at the same overhead, the proposed GURC and the conventional URC result in SER of $2.28 \times 10^{-4}$ and $2.30 \times 10^{-3}$, respectively. Therefore, in Fig. 6, it is noticed that we can further improve SERs against conventional URC where their parameters were optimized in [3] with the fixed degree distribution $\Phi_{\mathrm{conv}}(x)$. Performances for $K = 1000$ are also shown in Fig 6. Despite the degraded SERs, the proposed GURC outperforms the conventional URC.

We also evaluate the performances for the $u = 3$ case where the fractions of three classes are $\pi_1 = 0.1$, $\pi_2 = 0.2$, and $\pi_3 = 0.7$ with corresponding importance levels of $I_1 = \frac{10000}{10101}$, $I_2 = \frac{100}{10101}$, and $I_3 = \frac{1}{10101}$. Hence, the symbols within $A_1$ are the most important whereas those within $A_3$ are the least important among all the symbols. Fig. 7 shows the SERs of the three classes for the proposed and the conventional URC. Since no optimal symbol-selection weight has been investigated for $u \geq 3$, the conventional URC employs the same symbol-selection weights as the proposed scheme for comparison. As a result, the proposed GURC outperforms the conventional URC for all classes, $A_1$, $A_2$, and $A_3$. For $A_1$, the proposed GURC and the conventional URC result in SERs of $8.89 \times 10^{-7}$ and $9.02 \times 10^{-5}$, respectively, at the overhead $\gamma_{\mathrm{gurc}} = 1.15$. For $A_2$, the proposed GURC and the conventional URC result in SERs of $2.5 \times 10^{-5}$ and $1.33 \times 10^{-3}$, respectively, at the same overhead. For $A_3$, the proposed GURC and the conventional URC result in SERs of $2.7 \times 10^{-4}$ and $2.21 \times 10^{-3}$, respectively.

For the distributed wireless relay network we assumed in this paper, two kinds of rateless codes are used by the source and relays. By employing the proposed GDRC and GURC, we can guarantee optimal performances for the S-R and R-D channels, as shown in Figs. 5, 6, and 7. With the optimal rateless codes, the end-to-end performances for the distributed wireless relay networks were evaluated. The end-to-end performance with the optimal GDRC was achievable only when all the information about common symbols is available, which is an ideal network case. Let the network contain three relays, $M = 3$, and the erasure rates of the S-R channels be $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.2$. Figs. 8 shows the end-to-end SERs with the URC optimal for $u = 2$ and $u = 3$, respectively. The message length, encoding overhead at the source and relays are $K = 10000$, $\gamma_{\mathrm{gurc}} = 1.15$, and $\gamma_{\mathrm{gdrc}} = 1.1$, respectively. Since there are two options for both of the source and relays, respectively, the performances are compared for four possible cases.

We note a similar tendency from both Fig. 8. When we observe from case I to II, or from case III to IV, it is clear that the optimal GDRC help the second decoder for URC to retain as many SO symbols as possible. If all the information about common symbols is known at the second hop, we can achieve the performances of case I and III with the proposed GURC and the conventional URC, respectively. In other words, case I and III show us the end-to-end SERs for an ideal network. In addition, when we compare case I and III, or cases II and IV, it is shown that the proposed GURC outperforms the conventional URC with the same amount of SO symbols. Since the proposed rateless codes achieve superior performances for both the S-R and R-D channels, case I shows the most improved performance, whereas case IV shows the worst performance.

## VII. Conclusion

In this paper, generalized UEP rateless codes (GURC) and generalized distributed rateless codes (GDRC) were proposed for distributed relay networks that contain a single source, multiple relays, and a single destination. We presented symbol error rates of GURC and GDRC to show that we can obtain a substantial performance gain over the conventional scheme. Particularly in case of GDRC, it is an open problem to design distributed rateless coding scheme that achieve the performance of the optimal GDRC with the minimum amount of information of common symbols. Nevertheless, novel optimization methods using SQP were proposed to design an optimal GURC and GDRC for cases that the conventional scheme could not solve. As a result, it became possible to recover all the message symbols via a two-step BP decoding with high probability, regardless of the number of required importance classes or the number of relays in the system. Consequently, we could provide the fully generalized design tool for UEP rateless codes as well as we could estimate achievable end-to-end performances by the optimal GDRC.

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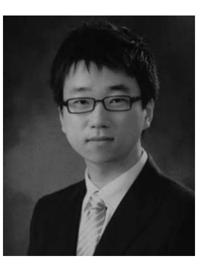

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