Distributed Unequal Error Protection Rateless Codes over Erasure Channels: A Two-Source Scenario

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***Abstract*—In *distributed rateless* coding, multiple disjoint sources need to deliver their *rateless* coded symbols (where a symbol may contain a single bit or thousands of bits) to a *common* destination via a single *relay*. In this paper, we propose and design novel distributed rateless codes called *DU-rateless codes* that can provide *Unequal Error Protection* (UEP) for disjoint sources with *unequal* data lengths on erasure channels. To design DU-rateless codes, we tune the coding parameters at each source and propose to *smartly* combine the encoded symbols at the relay.**

We analyze DU-rateless codes employing *And-Or tree analysis* technique and leverage our analysis to design several sets of codes for various setups employing the-state-of-the-art *multi-objective genetic algorithms*. We evaluate the performance of the designed codes using numerical simulations and discuss their advantages.

***Index Terms*—Distributed rateless codes, genetic algorithms optimization, unequal error protection, erasure channels.**

I. Introduction

LT CODES [1] are the first practical implementation of a class of modern *forward error correction* (FEC) codes referred to by *rateless codes*. The properties of an LT code is fully determined by its *degree distribution* called *Robust-Soliton* distribution [1]. The Robust-Soliton distribution is carefully designed to achieve a capacity-approaching performance on *erasure channels* [1].

However, LT codes did not target *distributed* data collection; hence they may suboptimally perform in distributed data collection [2]. In distributed data collection r data sources need to transmit their rateless encoded symbols to a common destination through a single *relay*. For instance, r nodes within a *cluster* in a *wireless sensor network* (WSN) that transmit their rateless coded data to a *base station* via a *cluster-head* form such a distributed data collection. It is worth noting that r sources may have different data block lengths and different data *importance levels*. Consequently, we are interested in designing flexible *distributed rateless codes* that in general can provide *Unequal Error Protection* (UEP) for data of *various lengths*.

In this paper, as a first step we consider $r = 2$ and propose *distributed UEP rateless codes* (DU-rateless codes), which are a realization of such codes on erasure channels. In DU-rateless codes *input symbol* length can be arbitrary from one-bit (binary) symbol to hundreds or thousands of bits similar to LT codes. The problem in DU-rateless codes is to tune a degree distribution for each source and to design relaying parameters to achieve (almost) minimal decoding error rates with a certain ratio referred to by *UEP gain*. Similar to LT codes, DU-rateless codes are also *universal* [1] meaning that they are simultaneously near optimal for every erasure channel.

We employ And-Or tree analysis technique to study DU-rateless codes and utilize our analytical results to design several *close to* optimal DU-rateless codes for various setups employing a *multi-objective genetic algorithm* called *NSGA-II* [3]. Finally, we report the designed codes and evaluate their performance. This paper extends our initial results on DU-rateless codes that appeared in [4].

The paper is organized as follows. In Section II, we review related work. In Section III, we propose and analyze DU-rateless codes. In Section IV, we employ NSGA-II to design DU-rateless codes. Further, in Section V we evaluate the performance of several ensembles of DU-rateless codes. Finally, Section VI reports the future work and concludes the paper.

II. Related Work

Authors in [2] have designed *distributed LT* (DLT) codes. In DLT coding, Robust-Soliton distribution is decomposed into r identical distributions to encode input symbols at r sources. Next, the encoded symbols are selectively combined or forwarded with certain probabilities to the destination such that the delivered coded symbols follow Robust-Soliton degree distribution (which is known to be capacity-achieving).

Authors in [5], considered rateless coding at r sources with an identical degree distribution. In [5], the number of combined encoded symbols (regardless of their degree) at the relay is determined by a second independent degree distribution. Authors have analyzed their codes and designed a few distributed rateless codes. In [6] authors considered a network with two sources $r = 2$ and designed a simple forwarding from the relay such that the degree distribution of the delivered symbols to destination follows a *Soliton-like* distribution (SLRC). Authors have shown that SLRC codes outperform DLT codes. Further, SLRC codes reduce to LT codes when a source leaves.

Authors in [7] propose an *online* encoding ensemble of LT codes such that the i^{th} output symbol is strictly comprised

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of the first i input symbols. They design their encoding and relaying scheme such that delivered symbols to destination maintain Robust-Soliton distribution. The scheme proposed in [7] may not be distributively implemented in contrast to DU-rateless codes. Authors in [8], [9], proposed *UEP* rateless codes. Although codes designed in [8], [9] are capable of providing UEP, they may not be distributively implemented.

Therefore, we propose DU-rateless codes that are inspired by UEP rateless codes [8], [9], which are able to provide UEP for disjoint data blocks of unequal lengths. Further, we *jointly* optimize DU-rateless codes parameters to obtain an optimal coding performance. The comparable scheme to DU-rateless codes is employing an independent LT codes at each source.

III. Distributed Unequal Error Protection Rateless Codes

In this section, we describe DU-rateless coding/decoding.

A. Proposed Coding and Decoding

Consider a distributed data collection with two sources s_1 and s_2 with data block of lengths ρk and k *input* symbols, respectively, where $0 < \rho \leq 1$. Let S_1 and S_2 denote the set of s_1 and s_2 input symbols, respectively. Note that without loss of generality and for simplicity we assume that the symbols are binary symbols.

To generate a rateless coded *output* (encoded) symbol from k input symbols, first its degree is randomly chosen to be d with probability Ω_d using a degree distribution $\{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ (also shown by its generator polynomial $\Omega(x) = \sum_{i=1}^{k} \Omega_i x^i$). Next, d input symbols are selected *uniformly at random* and are bitwise *XOR*ed to form the output symbol. We call the d contributing input symbols in forming an output symbol as its *neighbors*. $\Omega(x)$ is carefully optimized so that k input symbols can be recovered from $k\gamma_{succ}$ output symbols with a high probability, where γ_{succ} is called the *coding overhead* and asymptotically ($k \to \infty$) approaches 1. However, for practical finite values of k, γ_{succ} may be much larger than 1.

In DU-rateless coding, s_1 employs $\Omega(x)$ to encode its data block S_1 (in the same way that input symbols are encoded by Robust-Soliton distribution in LT coding). Similarly, s_2 employs $\varphi(x)$ to encode S_2. Next, s_1 and s_2 transmit their output symbols to a common relay R, which based on the following two rules generates three types of output symbols and forwards them to a destination D.

1) With probabilities p_1 and p_2 it directly forwards s_1 and s_2's output symbols to D, respectively.
2) With probability $p_3 = 1 - p_1 - p_2$ it forwards the XOR of two incoming coded symbols to D.

The decoding process of LT and DU-rateless codes are identical and is performed iteratively as follows. Find an output symbol such that the value of all but one neighboring input symbol is known. Recover the value of the unknown input symbol by bitwise XOR operations. Repeat this process until no such an output symbol exists. As we later show, iterative decoding of rateless codes is a form of *belief propagation* decoding. The DU-rateless decoding succeeds with a high probability when $(1+\rho)\gamma_{succ}k$ output symbols are received at D. For a received coding overhead of $0 \leq \gamma \leq \gamma_{succ}$, the proposed DU-rateless code ensemble is specified by parameters $(\rho k, k, \Omega(x), \varphi(x), p_1, p_2, p_3, \gamma)$.

Let ε_1, ε_2, and ε_3 denote the erasure rates of $s_1 - R$, $s_2 - R$, and $R - D$ channels, respectively. Further, assume packet transmission at s_1 and s_2 is not synchronized. With this setup, we need to set the *symbol transmission rates* of s_1 and s_2 such that no huge symbol buffering or dropping is required at R. It is not hard to show that s_2 needs to generate $\frac{(1-p_1)(1-\varepsilon_2)}{(1-p_2)(1-\varepsilon_1)}$ output symbols per one output symbol generated at s_1 so that in expectation no symbols are buffered. We should note that due to random losses of s_1 and s_2 symbols and their asynchronous transmissions, R may need to buffer only a few symbols for a short period time. For example, assume R decides to combine s_1 and s_2 symbols. However, due to random losses on the channel several symbols from s_1 arrive while no symbols from s_2 arrives. In such a case, R needs to buffer a few symbols from s_1 until symbols from s_2 arrive. Therefore, the transmission rate of $\frac{(1-p_1)(1-\varepsilon_2)}{(1-p_2)(1-\varepsilon_1)}$ symbol at s_2 guarantees that R may have to buffer only a few symbols for a short period of time.

B. And-Or Tree Analysis of the Proposed Codes

To investigate the recovery probability of an input symbol in DU-rateless decoding on erasure channels, we extend the And-Or tree analysis [10], [11] technique to fit the decoding process of DU-rateless codes. The input and output symbols of a DU-rateless code can be viewed as vertices of a bipartite graph G, where the input symbols are the *variable nodes* and the output symbols are the *check nodes* [12], [13]. In DU-rateless coding, the corresponding bipartite graph at the receiver has two types of variable nodes (mapped to S_1 and S_2), and three types of check nodes generated by R. Let C_1 and C_2 denote the set of output symbols directly forwarded from R, and C_3 denote the set of combined output symbols (see [4, Fig. 2]).

Clearly, C_1 symbols are generated based on $\Omega(x)$ and are only connected to S_1. Similarly, C_2 symbols are generated based on $\varphi(x)$ and are only connected to S_2. Finally, input symbols of C_3 are generated using both S_1 and S_2 with a degree distribution equal to $\Omega(x) \times \varphi(x)$ [2]. It is worth noting that the ratio of the number of symbols in C_1, C_2, and C_3 is equal to p_1, p_2, and p_3, respectively.

Let us choose $T_{l,1}$ a subgraph of G as following. Choose an edge (v, w) uniformly at random from all edges in G with one end among S_1 symbols. Call the input symbol v connected to edge (v, w) the root of $T_{l,1}$, which is assumed to be at depth 0. $T_{l,1}$ is a graph induced by v and all neighbors of v within distance $2l$ after removing the edge (v, w). It can be shown that $T_{l,1}$ is a *tree* asymptotically [8]–[10]. Similarly, we define $T_{l,2}$ such that the root of $T_{l,2}$ resides in S_2 symbols.

In addition, in the iterative belief propagation LT decoding process on binary erasure channels (BEC) we can assume that messages (0 or 1) are sent along the edges from output symbols to input symbols, and then vice-versa [8], [9], [11], [14]. An input symbol sends 0 to an adjacent output symbol if and only if its value is not recovered yet. Similarly, an output symbol sends 0 to an adjacent input symbol if and only if it is not able to recover the value of the input symbol. In other words, an input symbol sends 1 to a neighboring output symbol if and only if it has received at least one message with value 1 from other neighboring output symbol, hence it is performing the logical OR operation. Also an input symbol sends 0 to a neighboring output symbol if only if it has received at least one message with value 0 from its other neighboring input symbols, which is a logical AND operation. Therefore, $T_{l,1}$ and $T_{l,2}$ are And-Or trees with OR and AND nodes on even and odd depths, respectively (for pictorial illustration see [4, Fig. 3 and Fig. 4]). Note that we denote the symbols at depth $i+1$ as the children of symbols at depth i.

Let $\delta_{i,1}, i \in \{0,\ldots,A_1\}$ be the probability that an input symbol in S_1 has i children in C_1 or C_3. Further, let $\delta_{i,2}$ be the probability that a S_2 symbol has $i \in \{0,\ldots,A_2\}$ children in C_2 or C_3. Moreover, let C_1 symbols choose to have $i \in \{0,\ldots,B_1-1\}$ children from S_1 with probability $\beta_{i,1}$, and C_2 choose to have $i \in \{0,\ldots,B_2-1\}$ children from S_2 with probability $\beta_{i,2}$.

Moreover, in $T_{l,1}$ C_3 symbols choose $i \in \{0,\ldots,B_1-1\}$ and $j \in \{1,\ldots,B_2\}$ children from S_1 and S_2 symbols with probabilities $\beta_{i,1}$ and $\beta_{j,3}$, respectively. Further, in $T_{l,2}$, C_3 symbols can choose $i \in \{0,\ldots,B_2-1\}$ and $j \in \{1,\ldots,B_1\}$ children from S_2 and S_1 symbols with probabilities $\beta_{i,2}$ and $\beta_{j,4}$, respectively. The probabilities that the root input symbol of And-Or trees $T_{l,1}$ and $T_{l,2}$ evaluate to 0 is given in the following Theorem.

Theorem 1: Let $y_{l,1}$ and $y_{l,2}$ be the probabilities that the roots of the And-Or trees $T_{l,1}$ and $T_{l,2}$ evaluate to 0, respectively. Then we have

$$y_{l,1} = \delta_1\Big(1 - p'_1\beta_1(1-y_{l-1,1}) - p'_3 \sum_{i=1}^{B_1+B_2-1}\sum_{j=0}^{i-1}\left[\beta_{j,1}(1-y_{l-1,1})^j\beta_{i-j,3}(1-y_{l-1,2})^{i-j}\right]\Big), \tag{1}$$

$$y_{l,2} = \delta_2\Big(1 - p'_2\beta_2(1-y_{l-1,2}) - p'_4 \sum_{i=1}^{B_1+B_2-1}\sum_{j=0}^{i-1}\left[\beta_{j,2}(1-y_{l-1,2})^j\beta_{i-j,4}(1-y_{l-1,1})^{i-j}\right]\Big), \tag{2}$$

with $y_{0,1} = y_{0,2} = 0$, $\delta_1(x) = \sum_{i=0}^{A_1}\delta_{i,1}x^i$, $\delta_2(x) = \sum_{i=0}^{A_2}\delta_{i,2}x^i$, $\beta_1(x) = \sum_{i=0}^{B_1-1}\beta_{i,1}x^i$, $\beta_2(x) = \sum_{i=0}^{B_2-1}\beta_{i,2}x^i$, $p'_1 = \frac{p_1}{1-p_2}$, $p'_3 = \frac{1-p_1-p_2}{1-p_2} = \frac{p_3}{1-p_2}$, $p'_2 = \frac{p_2}{1-p_1}$ and $p'_4 = \frac{1-p_1-p_2}{1-p_1} = \frac{p_3}{1-p_1}$.

Proof: Consider output symbols of depth 1 in $T_{l,1}$ (which are of type C_1 and C_3). A C_1 symbol has children in S_1 symbols of depth 2* and evaluates to 1 with probability $\sum_{i=0}^{B_1-1}\beta_{i,1}(1-y_{l-1,1})^i$. A C_3 symbol may have between 0 to B_1-1 children from S_1 symbols and between 1 to B_2 children from S_2 symbols. Hence, the probability that such an input symbol evaluates to 0 is $\sum_{i=1}^{B_1+B_2-1}\sum_{j=0}^{i-1}[\beta_{j,1}(1-y_{l-1,1})^j\beta_{i-j,3}(1-y_{l-1,2})^{i-j}]$.

From the children of the *root* of $T_{l,1}$ at depth 0, p'_1 fraction are C_1 symbols and the rest p'_3 fraction are C_3 symbols. Hence, the probability that an output symbol that is a child of $T_{l,1}$'s root evaluates to 0 is $\Big(1 - p'_1\sum_{i=0}^{B_1-1}\beta_{i,1}(1-y_{l-1,1})^i - p'_3\sum_{i=1}^{B_1+B_2-1}\sum_{j=0}^{i-1}\left[\beta_{j,1}(1-y_{l-1,1})^j\beta_{i-j,3}(1-y_{l-1,2})^{i-j}\right]\Big)$.

Therefore, the probability that the root of $T_{l,1}$ evaluates to 0, $y_{l,1}$, is given by (1). Note that $y_{l,2}$ can be analyzed in a similar way to obtain (2). ∎

*Note that S_1 and S_2 symbols at depth 2 in $T_{l,1}$ (as well as in $T_{l,2}$) are the roots for independent And-Or tree $T_{l-1,1}$ and $T_{l-1,2}$, respectively.

To complete DU-rateless codes analysis, we only need to compute the probabilities $\beta_{i,1}$, $\beta_{i,2}$, $\beta_{i,3}$, $\beta_{i,4}$, and functions $\delta_1(x) = \sum_i \delta_{i,1}x^i$ and $\delta_2(x) = \sum_i \delta_{i,2}x^i$. First, we need to investigate the degree distribution of input symbols in S_1 and S_2. In the following lemma, we show that the degree (the number of edges connected to) of each input symbol in the proposed ensemble of DU-rateless code with parameters $(\rho k, k, \Omega(x), \varphi(x), p_1, p_2, p_3, \gamma)$ is Poisson-distributed asymptotically.

Lemma 1: Consider two sources s_1 and s_2 employing a $(\rho k, k, \Omega(x), \varphi(x), p_1, p_2, p_3, \gamma)$ DU-rateless code. Asymptotically, for a total received overhead of γ the degree of S_1 and S_2 input symbols in the corresponding bipartite graph G follow Poisson distributions with means $\lambda_1 = \Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}$ and $\lambda_2 = \varphi'(1)\gamma(1-p_1)(1+\rho)$, respectively.

Proof: The average degrees of $\Omega(x)$ and $\varphi(x)$ are given by $\sum_i i\Omega_i = \Omega'(1)$ and $\sum_i i\varphi_i = \varphi'(1)$, respectively. S_1 symbols are chosen based on $\Omega(x)$ and are included in a fraction $(1-p_2)$ of $(1+\rho)\gamma k$ total output symbols. Therefore, $\Omega'(1)(1+\rho)\gamma k(1-p_2)$ edges are connected uniformly at random to S_1 symbols. Consequently, a S_1 symbols has degree d with probability

$$\tau_{d,1} = \binom{(1-p_2)\Omega'(1)\gamma k(1+\rho)}{d} \times \left(\frac{1}{\rho k}\right)^d\left(1-\frac{1}{\rho k}\right)^{(1-p_2)\Omega'(1)\gamma k(1+\rho)-d}. \tag{3}$$

Similarly, $(1-p_1)\varphi'(1)k(1+\rho)\gamma$ edges are connected uniformly at random to S_2 symbols. As a result, a S_2 symbol has degree d with probability

$$\tau_{d,2} = \binom{(1-p_1)\varphi'(1)\gamma k(1+\rho)}{d} \times \left(\frac{1}{k}\right)^d\left(1-\frac{1}{k}\right)^{(1-p_1)\varphi'(1)\gamma k(1+\rho)-d}. \tag{4}$$

Asymptotically, (3) and (4) approach to

$$\tau_{d,1} = \frac{e^{-(1-p_2)\Omega'(1)\gamma\frac{(1+\rho)}{\rho}}\left[\Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}\right]^d}{d!}, \tag{5}$$

and

$$\tau_{d,2} = \frac{e^{-(1-p_1)\varphi'(1)\gamma(1+\rho)}\left[\varphi'(1)\gamma(1-p_1)(1+\rho)\right]^d}{d!}, \tag{6}$$

respectively, which are Poisson distributions with the means $\lambda_1 = \Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}$ and $\lambda_2 = \varphi'(1)\gamma(1-p_1)(1+\rho)$. ∎

Next, employing Lemma 1 we find $\beta_{i,1}$, $\beta_{i,2}$, $\beta_{i,3}$, $\beta_{i,4}$, $\delta_1(x) = \sum_i \delta_{i,1}x^i$, and $\delta_2(x) = \sum_i \delta_{i,2}x^i$ as a function of a DU-rateless code parameters in the following lemma.

Lemma 2: The probabilities $\beta_{i,1}$, $\beta_{i,2}$, $\beta_{i,3}$, $\beta_{i,4}$, and functions $\delta_1(x)$ and $\delta_2(x)$ for a

$(\rho k, k, \Omega(x), \varphi(x), p_1, p_2, p_3, \gamma)$ DU-rateless code are given as

$$\delta_1(x) = e^{(1-p_2)\Omega'(1)\gamma\frac{(1+\rho)}{\rho}(x-1)}, \delta_2(x) = e^{(1-p_1)\varphi'(1)\gamma(1+\rho)(x-1)},$$
$$\beta_{i,1} = \frac{(i+1)\Omega_{i+1}}{\Omega'(1)}, \text{ hence } \beta_1(x) = \frac{\Omega'(x)}{\Omega'(1)},$$
$$\beta_{i,2} = \frac{(i+1)\varphi_{i+1}}{\varphi'(1)}, \text{ hence } \beta_2(x) = \frac{\varphi'(x)}{\varphi'(1)},$$
$$\beta_{i,3} = \varphi_i, \text{ and } \beta_{i,4} = \Omega_i.$$

Proof: We have $\beta_{i,1}$ is the probability that a randomly chosen edge with one end in S_1 is connected to a C_1 or C_3 symbol with i children in S_1. Therefore, $\beta_{i,1}$ is the probability that a randomly selected edge with one end connected to a S_1 symbol has the other end connected to an output symbol in C_1 or C_3 with $(i+1)$ children in S_1. Therefore, we have $\beta_{i,1} = \frac{(i+1)\Omega_{i+1}}{\Omega'(1)}$ or equivalently $\beta_1(x) = \frac{\Omega'(x)}{\Omega'(1)}$, which is edge degree distribution from C_1 perspective. Similarly, we have $\beta_{i,2} = \frac{(i+1)\varphi_{i+1}}{\varphi'(1)}$, which gives $\beta_2(x) = \frac{\varphi'(x)}{\varphi'(1)}$, which is edge degree distribution from C_2 perspective.

Moreover, $\beta_{i,3}$ is the probability that a randomly chosen edge with one end in S_1 is connected to a C_3 symbol with i children in S_2. Therefore, $\beta_{i,3}$ is the probability that a randomly selected edge connected to a S_1 symbol in the graph G is connected to a C_3 output symbol with i children in S_2. This simply gives $\beta_{i,3} = \varphi_i$. In the same way, $\beta_{i,4} = \Omega_i$.

Further, we have $\delta_{i,1}$ is the probability that the input symbol connected to a randomly selected edge has degree $i+1$ given that the input symbol belongs to S_1. Therefore, $\delta_{i,1} = \frac{(i+1)\lambda_{i+1,1}}{\sum_i i\lambda_{i,1}}$, where $\lambda_{i,1}$ is given in Lemma 1. Using Lemma 1, we have

$$\delta_{i,1} = \frac{(i+1)\lambda_{i+1,1}}{\Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}},$$
$$= \frac{(i+1)e^{-(1-p_2)\Omega'(1)\gamma\frac{(1+\rho)}{\rho}}\left[\Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}\right]^{i+1}}{\Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}(i+1)!},$$
$$= \frac{e^{-(1-p_2)\Omega'(1)\gamma\frac{(1+\rho)}{\rho}}\left[\Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}\right]^{i}}{i!}.$$

After substitution, we have

$$\delta_1(x) = \sum_i \delta_{i,1}x^i,$$
$$= \sum_i \frac{e^{-(1-p_2)\Omega'(1)\gamma\frac{(1+\rho)}{\rho}}\left[\Omega'(1)\gamma(1-p_2)\frac{(1+\rho)}{\rho}x\right]^{i}}{i!},$$
$$= e^{(1-p_2)\Omega'(1)\gamma\frac{(1+\rho)}{\rho}(x-1)}.$$

Similarly, we have $\delta_2(x) = e^{(1-p_1)\varphi'(1)\gamma(1+\rho)(x-1)}$. ∎

Similar to [8, Lemma 4], we can show that the sequences $\{y_{l,1}\}_l$ and $\{y_{l,2}\}_l$ are monotone decreasing and are bounded in $[0,1]$, and they converge to fixed points. Let BER_1 and BER_2 denote the corresponding fixed points. These fixed points are the probabilities that S_1 and S_2 symbols are not recovered after l decoding iterations. In other words, these fixed points are the final decoding error rates of a $(\rho k, k, \Omega(x), \varphi(x), p_1, p_2, p_3, \gamma)$ DU-rateless code. To realize almost minimal BER_1 and BER_2, we will design DU-rateless codes with parameters that are *jointly* optimized for a given γ_{succ} in the next section.

IV. Distributed Unequal Error Protection Rateless Codes Design

For an ensemble of DU-rateless code with parameters $(\rho k, k, \Omega(x), \varphi(x), p_1, p_2, p_3, \gamma)$, we define the UEP gain $\eta \triangleq \frac{\text{BER}_2}{\text{BER}_1}$, where BER_1 and BER_2 can be computed from (1) and (2), respectively, for a large enough l. A larger η shows a higher recovery rate of S_1 input symbols at D or equivalently a higher level of protection compared to S_2. It is worth noting that $\eta = 1$ corresponds to *equal error protection* (EEP) case where S_1 and S_2 are equally protected. The question that arises is that what are the appropriate parameters $\Omega(x)$, $\varphi(x)$, p_1, p_2, and p_3 that would result in a *desired* η and minimal BER_1 and BER_2. It is not hard to show that BER_1 and BER_2 are two *conflicting* objective functions by investigating (1) and (2) (improving one may deteriorate the other one). Therefore, we have a *multi-objective optimization* problem.

A. Multi-Objective Optimization Genetic Algorithms

Let U and $\underline{u}$ denote the decision space and a decision vector, respectively, of an optimization problem. Let $F_1(\underline{u}), F_2(\underline{u}), \ldots, F_n(\underline{u})$ denote the conflicting objective functions. The problem is to find decision vectors $\underline{u}$ that *concurrently* minimize/maximize *all* objective functions. In a simple case with a single objective function, the problem boils down to a conventional minimization/maximization problem.

For a minimization problem, $\underline{u}_1 \in U$ is said to be dominated by $\underline{u}_2 \in U$, or $\underline{u}_1 \prec \underline{u}_2$, if $\forall\, i \in \{1, \ldots, n\},\ F_i(\underline{u}_1) \geq F_i(\underline{u}_2)$ and for at least one i, $F_i(\underline{u}_1) > F_i(\underline{u}_2)$. A non-dominated *pareto front* vector, $\underline{u}^*$, is a decision vector that no other decision vector can dominate. In other words, in a minimization problem no other decision vector exists such that it would decrease some objective functions without deteriorating at least one other objective function compared to $\underline{u}^*$.

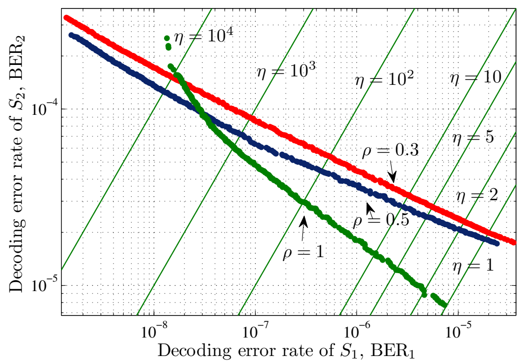
The set of all dominant solution vectors form *pareto optimal set*. The plot of objective functions of pareto optimal members in the objective space builds the *pareto front*. For a pictorial illustration of pareto front see [4, Fig. 5]

Multi-objective optimization methods search to find decision variables that result in pareto front members that are *well spread* and *equally spaced* to cover the whole pareto front. NSGA-II [3] is one of the many such algorithms with an outstanding performance that we employ in our design. Note that although genetic-algorithms have very high complexity, the optimization can be performed in an off-line mode and stored and the appropriate codes can be later selected based on the system requirements.

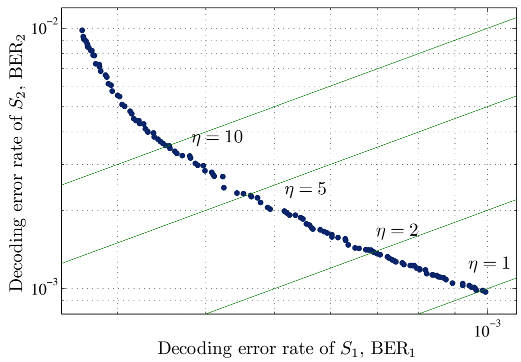
B. Proposed Codes Design Employing NSGA-II

We fix $\gamma_{succ} = 1.05$ and employ NSGA-II [3] to find the optimum $\Omega(x)$, $\varphi(x)$, p_1, p_2, and p_3 that *concurrently minimize* BER_1 and BER_2 for various values of $\eta = \frac{\text{BER}_2}{\text{BER}_1}$ and $\rho \in \{0.3, 0.5, 1\}$. In other words, we have a problem including two objective functions given by (1) and (2) (BER_1 and BER_2), with 202 independent decision variables, i.e. $\underline{u} = \{\Omega_1, \Omega_2, \ldots, \Omega_{10^2}, \varphi_1, \varphi_2, \ldots, \varphi_{10^2}, p_1, p_2\}$.

The output of our optimization are 3 databases of *close to optimal* degree distributions for $\rho \in \{0.3, 0.5, 1\}$, each embracing a large number of DU-rateless codes parameters that

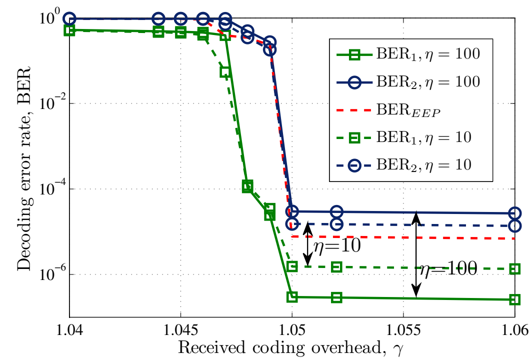


(a) The resulting pareto fronts for DU-rateless codes design with $\gamma_{succ} = 1.05$ and $\rho \in \{0.3, 0.5, 1\}$.

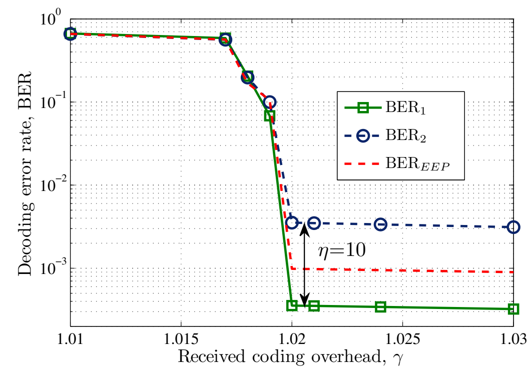


(b) The resulting pareto fronts for DU-rateless codes design with $\gamma_{succ} = 1.02$ and $\rho = 1$.

Fig. 1. The resulting pareto fronts for various DU-rateless codes setups



(a) The resulting BERs with optimized sets of parameters for $\eta \in \{10, 10^2\}$, $\gamma_{succ} = 1.05$, and $\rho = 1$.



(b) The resulting BERs with optimized sets of parameters for $\eta = 10$, $\gamma_{succ} = 1.02$, and $\rho = 1$.

Fig. 2. Asymptotic performance evaluation of the designed DU-rateless codes

realize various η's made available online at [15]. We emphasis that our results are close to optimal since genetic algorithms are known to find solutions that are not necessarily global-optimal but are rather *very close* to global-optimal solutions. In addition, confining the largest degree to $B_1 = B_2 = 10^2$ limits the degree distribution search space and results in the design of the codes that are suboptimal. Therefore, the performance of our designed DU-rateless codes is close to optimal. We plot the pareto fronts obtained from our optimizations in Fig. 1(a). Similarly, we set $\gamma_{succ} = 1.02$ and $\rho = 1$ and find the set of optimal DU-rateless codes for this setup with the pareto front illustrated in Fig. 1(b).

In Fig. 1 each point corresponds to two degree distributions and three relaying parameters $\Omega(x)$, $\varphi(x)$, p_1, p_2, and p_3. Fig. 1(a) shows that our designed DU-rateless codes are well spread with respect to η. One should choose an appropriate point according to a desired η and employ the corresponding DU-rateless code. From Fig. 1(b) we can see that due to much smaller γ_{succ} the minimum achievable error rates have increased, which shows an interesting trade-off between the achievable error-floor and the decoding overhead γ_{succ}. However, the UEP gain can be obtained for a wide range of η's.

V. Performance Evaluation of the Designed Codes

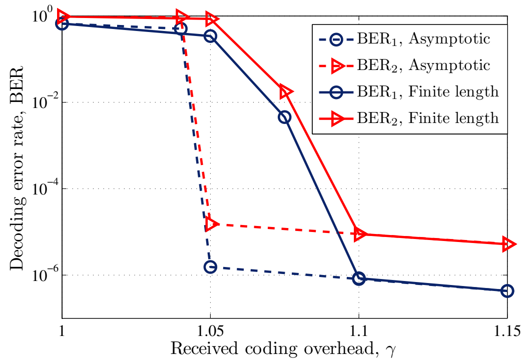
This section report the performance evaluation of our designed codes.

A. Asymptotic Performance Evaluation of the Designed Codes

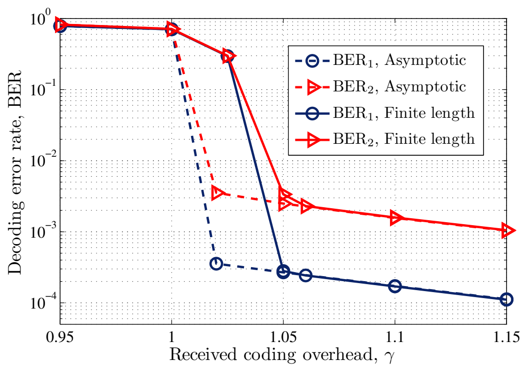
From the sets of our optimized DU-rateless codes available at [15], we choose two DU-rateless codes for $\eta \in \{10, 10^2\}$, $\rho = 1$, and $\gamma_{succ} = 1.05$ and evaluate their performance in Fig. 2(a) for $k \to \infty$ given by (1) and (2). For comparison, we have also plotted the BER_1 and BER_2 for EEP case ($\eta = 1$). Similarly, we choose an optimal DU-rateless codes with parameters $\gamma_{succ} = 1.02$ and $\rho = 1$ for $\eta = 10$ and evaluate its performance as shown in Fig. 2(b).

Fig. 2(a) shows that the expected UEP gain is fulfilled for $\gamma_{succ} = 1.05$ with the minimal values of BER_1 and BER_2. In addition, Fig. 2(b) shows that the expected UEP gain $\eta = 10$ is achieved although the error floors are higher due to smaller γ_{succ}. The parameters of a DU-rateless code for $\rho = 1$, $\eta = 10$, and $\gamma_{succ} = 1.05$ with performance illustrated in Fig. 2(a) is given as follows.

$$\begin{aligned}\Omega(x) =& 0.039x^1 + 0.492x^2 + 0.094x^3 + 0.09x^4 + 0.096x^5 + 0.002x^6 \\ &+ 0.055x^7 + 0.019x^8 + 0.033x^9 + 0.014x^{10} + 0.004x^{20} \\ &+ 0.005x^{27} + 0.001x^{28} + 0.004x^{31} + 0.001x^{39} + 0.005x^{43} \\ &+ 0.004x^{78} + 0.001x^{79} + 0.005x^{86} + 0.01x^{95} + 0.004x^{96} \\ &+ 0.001x^{99} + 0.006x^{100},\end{aligned} \tag{7}$$



(a) BER's for DU-rateless codes optimized for $\gamma_{succ} = 1.05$.



(b) BER's for DU-rateless codes optimized for $\gamma_{succ} = 1.02$.

Fig. 3. The resulting BERs for asymptotic case and finite length case ($k = 10^4$) for DU-rateless codes optimized for $\gamma_{succ} = 1.05$ and $\gamma_{succ} = 1.02$ with parameters $\eta = 10$ and $\rho = 1$.

$$\begin{aligned}\varphi(x) =& 0.072x^1 + 0.48x^2 + 0.055x^3 + 0.051x^4 + 0.063x^5 + 0.059x^6 \\ &+ 0.037x^7 + 0.026x^8 + 0.025x^9 + 0.036x^{10} + 0.005x^{15} \\ &+ 0.001x^{25} + 0.002x^{28} + 0.005x^{37} + 0.002x^{44} + 0.001x^{67} \\ &+ 0.001x^{70} + 0.001x^{76} + 0.001x^{77} + 0.002x^{83} + 0.001x^{84} \\ &+ 0.001x^{88} + 0.003x^{93} + 0.052x^{95} + 0.002x^{97},\end{aligned} \tag{8}$$

with $p_1 = 0.4822$, and $p_2 = 0.1173$, which gives $p_3 = 0.4005$. We can see that to achieve an optimum distributed coding 40.05% of the generated output symbols at s_1 and s_2 should be combined at the relay.

B. Performance Evaluation for Finite-length

Our designed DU-rateless codes are optimized based on our analytical results derived in Section III for asymptotic case, i.e., $k \to \infty$. However, in practice k is finite. Therefore, we set the parameters $\rho = 1$ and $\eta = 10$ for two cases of $\gamma_{succ} = 1.05$ and $\gamma_{succ} = 1.02$ and evaluate the performance of DU-rateless code for $k = 10^4$ using numerical encoding and decoding versus anosmatic setup as shown in Fig. 3. To find BER$_1$ and BER$_2$ in the finite length case, we take average over decoding error rates of 10^5 numerical decoding iterations. Fig. 3 shows that the expected UEP gain ($\eta = 10$) and minimal error rates are realized at slightly larger γ_{succ}'s. Therefore, our designed DU-rateless codes can indeed be employed for finite k cases as well for a larger γ_{succ}.

C. Performance Comparison with LT and DLT Codes

In this section, we compare the performance of DU-rateless codes with the case where s_1 and s_2 independently employ two LT codes $\mathcal{C}_1$ and $\mathcal{C}_2$ to generate C_1 and C_2, and R directly and intermittently forwards them to D. To perform the comparison, we set the parameters $k = 10^4$, $\rho = 1$, and $\eta = 10$. The DU-rateless code optimized for this setup has degree distributions given by (7) and (8) with $p_1 = 0.4822$, $p_2 = 0.1173$, and $p_3 = 0.4005$, which achieves BER$_1 \approx 5 \times 10^{-7}$ and BER$_2 \approx 5 \times 10^{-6}$ at $\gamma = 1.15$. This DU-rateless code results in output symbols with average degree of $\mu_{DU} \approx 11.38$.

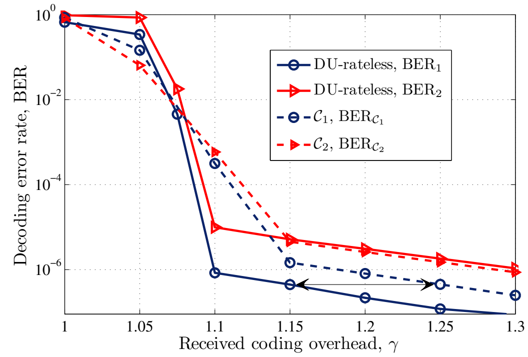


Fig. 4. Performance comparison of the employed DU-rateless code and the equivalent optimal separate LT codes. As shown, the overhead for achieving BER$_1 = 5 \times 10^{-7}$ reduces from 1.25 to 1.15 if we employ a DU-rateless code instead of two separate LT codes.

To perform a *fair* comparison, we need to have equivalent *decoding complexities* in both setups. Since the decoding complexity of LT decoding is determined by the average output symbols degree [1], we need to maintain the same average coded degree when two LT codes replace this DU-rateless code. Let $\mathcal{C}_1(c_1, \nu_1)$ and $\mathcal{C}_2(c_2, \nu_2)$ denote the desired LT codes, where c_1, ν_1, c_2, and ν_2 are the respective Robust-Soliton degree distributions parameters [1]. Further, assume that $\mathcal{C}_1(c_1, \nu_1)$ and $\mathcal{C}_2(c_2, \nu_2)$ have average output symbol degrees of $\mu_{\mathcal{C}_1}$ and $\mu_{\mathcal{C}_2}$ and realize the desired BER's at $\gamma_{\mathcal{C}_1}$ and $\gamma_{\mathcal{C}_2}$ in rateless decoding, respectively. Consequently, to have equal decoding complexities in both setups we need to find $\mathcal{C}_1(c_1, \nu_1)$ and $\mathcal{C}_2(c_2, \nu_2)$ such that $\frac{\gamma_{\mathcal{C}_1}\mu_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}\mu_{\mathcal{C}_1}}{\gamma_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}} = \mu_{DU}$. On the other hand, we should select c_1, ν_1, c_2, and ν_2 such that $\mathcal{C}_1(c_1, \nu_1)$ and $\mathcal{C}_2(c_2, \nu_2)$ can realize the desired BER's at minimum possible total overhead $\gamma_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}$. Therefore, to find $\mathcal{C}_1(c_1, \nu_1)$ and $\mathcal{C}_2(c_2, \nu_2)$ we solve the following minimization problem:

$$\begin{aligned}&\underset{c_1,\nu_1,c_2,\nu_2}{\operatorname{argmin}} \; (\gamma_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}) = [c_1^*, \nu_1^*, c_2^*, \nu_2^*], \\ &\text{s.t. } \frac{\gamma_{\mathcal{C}_1}\mu_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}\mu_{\mathcal{C}_2}}{\gamma_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}} = \mu_{DU}, \\ &\text{BER}_1 \leq 5 \times 10^{-7}, \text{and BER}_2 \leq 5 \times 10^{-6}.\end{aligned} \tag{9}$$

We search the whole decision space of c_1, ν_1, c_2, and ν_2 to find the global minimum of $\gamma_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2}$. The optimal $\mathcal{C}_1$ has parameters $c_1 = 0.1, \nu_1 = 40, \gamma_{\mathcal{C}_1} = 1.15$, and $\mu_{\mathcal{C}_1} = 10.74$. Further, the optimal $\mathcal{C}_2$ has parameters $c_1 = 0.1, \nu_1 = 15, \gamma_{\mathcal{C}_1} = 1.25$, and $\mu_{\mathcal{C}_1} = 11.98$. We have compared the performance of the setup with two separate LT codes $\mathcal{C}_1(c_1, \nu_1)$ and $\mathcal{C}_2(c_2, \nu_2)$ along with the equivalent DU-rateless code in Fig. 4.

Fig. 4 shows that the total amount of required overhead has decreased from $\gamma_{\mathcal{C}_1} + \gamma_{\mathcal{C}_2} = 2.4$ in separate coding setup to

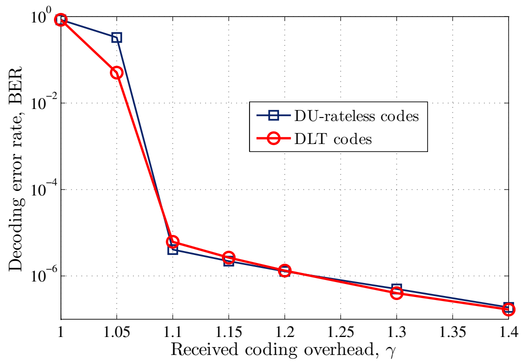


Fig. 5. Performance comparison of the DU-rateless codes for designed for $\rho = 1$, $\eta = 1$, $\gamma = 1.05$ and the DLT codes with average output degree of 11.03 for $k = 10^4$.

$(1+\rho)\gamma_{succ} = 2.3$ in the setup employing DU-rateless codes. This shows that when DU-rateless codes are employed 10^3 fewer symbols need to be delivered to receiver. Therefore, in our example DU-rateless codes can make 25% reduction in the number of required redundant received output symbol for successful decoding compared to two separate LT codes. This improvement is realized by increasing data block length, which is obtained by combining output symbols at the relay.

To compare DU-rateless codes with DLT codes [2], we have to select a DU-rateless code with $\rho = 1$ and $\eta = 1$ since DLT codes can only encode data blocks of equal size and may only provide EEP. This DU-rateless code results in the generation of output symbols with average degree of 11.03. Similar to comparison with regular LT codes, we find a Robust-Soliton distribution for DLT coding with average degree 11.03 and compare its performance to the selected DU-rateless code in Fig. 5 for $k = 10^4$. Fig. 5 interestingly shows that for $\rho = 1$ and $\eta = 1$ DLT and DU-rateless codes have almost the same performance and achieve the same error floor. However, we should note that DU-rateless codes are capable of providing UEP and also support sources with unequal block sizes.

VI. Conclusion

In this paper, we proposed *DU-rateless codes*, which are distributed rateless codes with *Unequal Error Protection* (UEP) property for two data sources with *unequal* data block lengths over erasure channels. First, we analyzed DU-rateless codes employing And-Or tree analysis technique, and then we designed several close to optimum sets of DU-rateless codes using multi-objective genetic algorithms. Performance comparison of the designed DU-rateless codes showed that they fulfilled the expected UEP property with almost minimal error rates. We also showed that although DU-rateless codes are designed for large message lengths, they can be employed for finite message lengths as well. Finally, we showed that DU-rateless codes surpass the performance of exiting codes for distributed rateless coding.

DU-rateless codes can be extended to networks with more than two sources ($r > 2$). In this case, there would be $2^r - 1$ relaying parameters to tune and our analysis can be extended for r sources, which is trivial but cumbersome; hence it is left for our future work. Further, DU-rateless codes can be designed for non-erasure channels such as AWGN channels employing multi-objective genetic algorithms, which is also left to future work.

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